Physics 740: Spring 2007: 04/14/07

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... and Traffic.

1. Nonlinear waves, some concluding remarks.

Most of the work we have done on nonlinear waves has focused on the Korteweg deVries equation, derived in Note 12, and on several *ad hoc* nonlinear equations, nonlinear Schroedinger, sine-Gordon, etc. The orientation of our interest was in the solitary waves (solitons) that could exist in the systems described by these equations.

Let's spend a minute with nonlinear effects in fluid dynamics. Where do they come from? The nonlinear effects come in principle from those terms that we drop when the equations of fluid dynamics are linearized. For simplicity take the ideal fluid having no transport phenomena.

1. In the continuity equation, Note 1, Eq. (1), the term neglected in Eq. (3) is

$$\nabla \cdot \rho \mathbf{v} - \rho_0 \nabla \cdot \mathbf{v} = \delta\rho \nabla \cdot \mathbf{v} + \nabla \delta\rho \cdot \mathbf{v}.. \tag{1}$$

2. In the Euler equation, Note 1, Eq. (2), there are two sources of nonlinearity

   (a) the intrinsic nonlinearity

$$(\mathbf{v} \cdot \nabla)\mathbf{v} \tag{2}$$

   (b) nonlinearity from the EOS used to *close* the equations, e.g.,

$$\delta P(\rho) = \left(\frac{\partial P}{\partial \rho}\right)_S \delta\rho + \left(\frac{\partial^2 P}{\partial \rho^2}\right)_S \frac{\delta\rho^2}{2} + \cdots \tag{3}$$

The nonlinearity in the EOS is usually the largest of these.

Equations (1) and (2), Note 1, can be re-arranged without approximation to read

$$\frac{\partial^2 \rho}{\partial t^2} - \nabla^2 P = \nabla \cdot [\rho\ (\mathbf{v} \cdot \nabla)\mathbf{v}] + \nabla \cdot [\mathbf{v}\ \nabla \cdot (\rho \mathbf{v})]. \tag{4}$$

When Eq. (3) is used we have

$$\frac{\partial^2 \rho}{\partial t^2} - c_S^2 \nabla^2 \rho = \frac{1}{2}\left(\frac{\partial c_S^2}{\partial \rho}\right)_S \nabla^2\ \rho^2 + \nabla \cdot [\rho\ (\mathbf{v} \cdot \nabla)\mathbf{v}] + \nabla \cdot [\mathbf{v}\ \nabla \cdot (\rho \mathbf{v})], \tag{5}$$

TABLE I: δP, Q and nonlinearity

source	δP	Q
threshhold of hearing	10^{-5}Pa	10^{-10}
normal speech	10Pa	10^{-4}
loud speech	10^{4}Pa	10^{-1}
rock concert	10^{5}Pa	1
jet take off	10^{9}Pa	10^{4}

where terms of order ρ^3, from Eq. (3) have been dropped. The EOS provides the first term on the RHS. Assume this is the largest nonlinearity and drop the two terms in $\rho\mathbf{vv}$. One of the things we want is some sense of how large this nonlinear term is, when is it important? Construct the parameter

$$\gamma_S = \frac{\rho_0}{c_S^2}\left(\frac{\partial c_S^2}{\partial \rho}\right)_S \tag{6}$$

and write Eq. (5) in the form (use $\delta\rho$ in place of ρ for emphasis)

$$\frac{\partial^2 \delta\rho}{\partial t^2} - c_S^2 \nabla^2 \delta\rho = c_S^2 \gamma_S \nabla^2 \, \frac{\delta\rho^2}{2\rho_0}. \tag{7}$$

Use $Q = \delta\rho/\rho_0$ a dimensionless measure of the size of the fluctuations in ρ. Then we have

$$\frac{\partial^2 Q}{\partial t^2} - c_S^2 \nabla^2 Q = c_S^2 \gamma_S \nabla^2 \, \frac{Q^2}{2}. \tag{8}$$

The quantity γ_S is often called the "B/A parameter" (look on google) and typically has a value between 1 and 10. If we write Eq. (8) in the form

$$\frac{\partial^2 Q}{\partial t^2} = c_S^2 \nabla^2 \left(1 + \frac{\gamma_S}{2} Q\right) Q, \tag{9}$$

then it is easy to judge the relative importance of the nonlinear term. For an ideal gas $\delta\rho/\rho_0 = \delta P/P_0$. Use this to judge the size of Q. Here are some numbers: Table I. For $5Q$ ($\gamma_S = 10$) as a measure of nonlinearity loud speech, a rock concert, jet take off will have important nonlinear aspects.

Fourier analysis. Perhaps the most important qualitative consequence of nonlnearity in wave equations is the *failure* of single frequency Fourier analysis. It is single frequency Fourier analysis that was used to find the dispersion relation for waves in an ideal fluid, in a fluid with viscosity and thermal conductivity, Note 1 and problem 3 on HW3. The nonlinear term Q^2 in Eq. (9) means that (1) if there is frequency ω in Q, e.g., $Q \propto exp - i\omega t$, the nonlinear term causes frequency 2ω ($Q^2 \propto exp - 2i\omega t$) to be a part of Q, (2) if there is frequency 2ω in Q, e.g., $Q \propto exp - 2i\omega t$, the nonlinear term causes frequency 4ω ($Q^2 \propto exp - 4i\omega t$) to be a part of Q, etc.

2. Advection and Traffic.

The continuity equation, under certain conditions, can be looked at without recourse to the Euler equation. When this is done it can be manipulated into the form of an advection equation and some aspects of solution can be learned by using methods appropriate to such equations. [The moves we make next are made to get to the equation we want quickly. The same equation arises in a more general case following a fairly detailed discussion, L and L Sec. 101.]

The continuity equation, in one dimension, is

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}\rho v = 0. \tag{10}$$

Suppose that we are deling with a material for which the velocity v is a known functiion of the density, i.e., there is an *EOS* of the form

$$v = v(\rho). \tag{11}$$

Italics around *EOS* because this is certainly no thermodynamic equation of state, v isn't even a thermodynamic variable.

[Begin Aside.] To give this notion meaning consider a particular example, a traffic model. Certainly cars and the density of cars on a highway obey the continuity equation (conservation of number or mass). Suppose the cars move according to the rules

1. the density can take on any value from 0 (you have the road to yourself) to ρ_m (traffic jam), $0 \leq \rho \leq \rho_m$.

2. the velocity of the density of cars at a place where the density is ρ is given by the *EOS*

$$v = v_m \left(1 - \frac{\rho}{\rho_m}\right). \tag{12}$$

[End Aside.]

Use Eq. (11) in Eq. (10) and write

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}\rho v(\rho) \;=\; 0, \tag{13}$$

$$\frac{\partial \rho}{\partial t} + \frac{d[\rho v(\rho)]}{d\rho}\frac{\partial \rho}{\partial x} \;=\; 0, \tag{14}$$

$$\frac{\partial \rho}{\partial t} + c(\rho)\frac{\partial \rho}{\partial x} \;=\; 0. \tag{15}$$

Here $F(\rho) = \rho v(\rho)$ is the flux and $c(\rho)$ is a velocity associated with the flux, $c = \partial F/\partial \rho$. Equations (13)- (15) are various forms of the **advection equation**; one variable, the density, and motion in space and time. These equations are among the simplest PDEs. [Even so they are not necessarily easy to handle numerically.]

The method of characteristics. The method of characteristics is first a way of getting an idea of what an advection equation is saying and second, depending on your fortitude, a way of solving such an equation. See Fig. 1. Suppose that at $t = 0$ the density ρ is a prescribed function of x. In the x-t plane from the t=0 axis at x_1 draw the line with slope $1/c(\rho_1)$ where ρ_1 is the density at $t = 0$ at x_1. Do the same at x_2, etc. The lines so drawn are called the *characteristics*. The density is unchanging along these lines. Why? Because $\rho(x, t)$ changes due to change in t and change in x as here

$$\frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + \frac{\partial \rho}{\partial x}\frac{dx}{dt}. \tag{16}$$

On a characteristic $dt/dx = 1/c(\rho)$ so that the equation for $d\rho/dt$ on the charateristic is

$$\frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + \frac{\partial \rho}{\partial x}c(\rho), \tag{17}$$

which is zero from Eq. (15). So the density on the $t = 0$ axis at x moves along the characteristic at x as time evolves. When the characteristics are an opening fan, like near $x = 0$, the characteristics dilute the density. When the characeristics are a closing fan, like near large x, the characteristics increase the density, they may even cross one another and appear to give ambiguous results. This situation can be handled by being careful. As we want the qualitative idea from the characteristic picture we won't worry about this. Basically *shock like* features may occur where the characteristics bring the density together. By *shock-like*

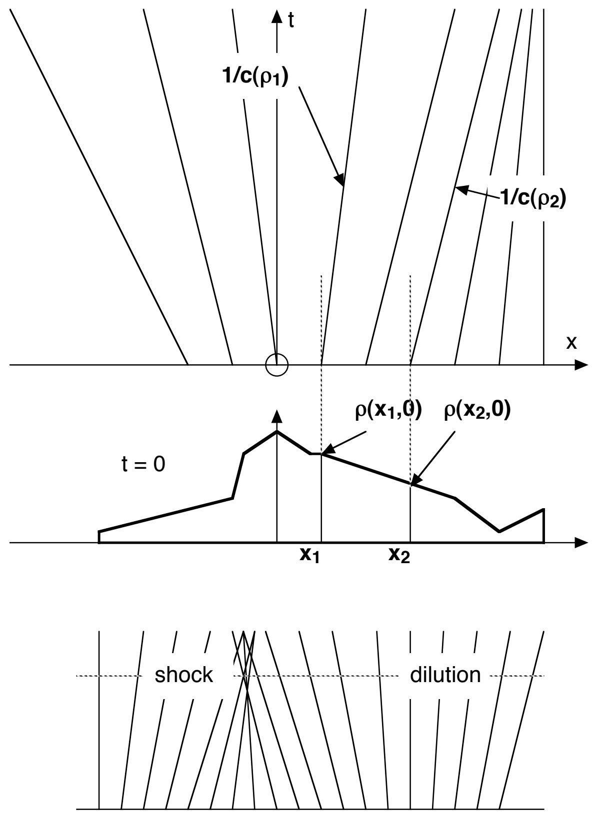

FIG. 1: Characteristics.

we mean that a discontinuity will develop in ρ as a function of x at fixed t. In a handout there were 4 examples of initial density profiles and the corresponding characteristics.

Traffic. We will follow the traffic example through a number of steps, Fig. 2. The initial density is that for traffic at a stop light, middle of figure. The characteristics for this density profile are shown at the top. At $t = 0$ the light changes to green. At a time t the first car is at $x = v_m t$, along the characteristic $t = x/v_m$. The characteristic from the origin for density ρ_m moves to the left along $t = -x/v_m$. A car in the pack can first move when this characteristic passes it. See the red car that stands still until this event at the bottom of the figure. At time t, bottom of Fig. 2, we can read off an equation for the density

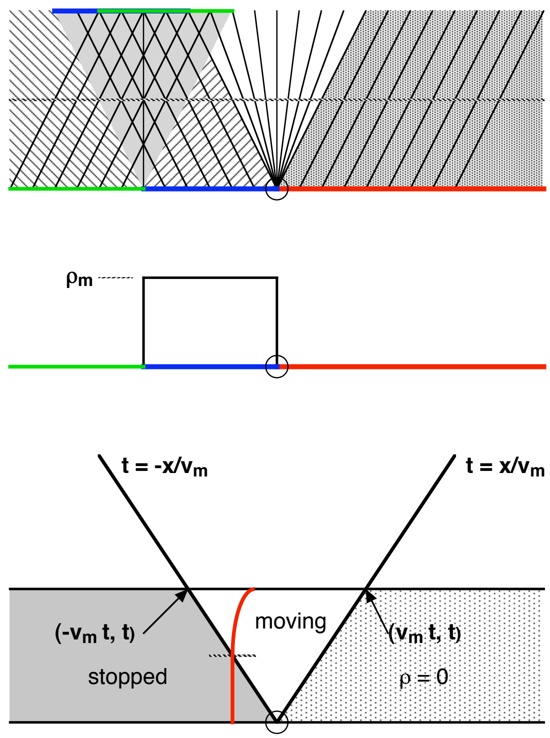

FIG. 2: Traffic problem.

$$\rho(x) = \rho_m, \quad t < -x/v_m, \tag{18}$$

$$\rho(x) = \frac{1}{2}\rho_m\left(1 - \frac{x}{v_m t}\right), \quad -v_m t < x < v_m t, \tag{19}$$

$$\rho(x) = 0, \quad v_m t < x. \tag{20}$$

From these equations we can find the position of a car as a function of time. A car moves among densities; it does not follow a single characteristic. We have $dx/dt = v(\rho_m) = 0, \quad t < -x/v_m$ and

$$\frac{dx}{dt} = v(\rho(x,t)) = \frac{1}{2}\rho_m\left(1 - \frac{x}{v_m t}\right), \quad t < -x/v_m. \tag{21}$$

For $t > t_0(x) = -x/v_m$ (remember the relevant values are x are at $x < 0$) the solution to this equation is

$$x(t) = v_m t - 2\sqrt{v_m t_0 v_m t}. \tag{22}$$

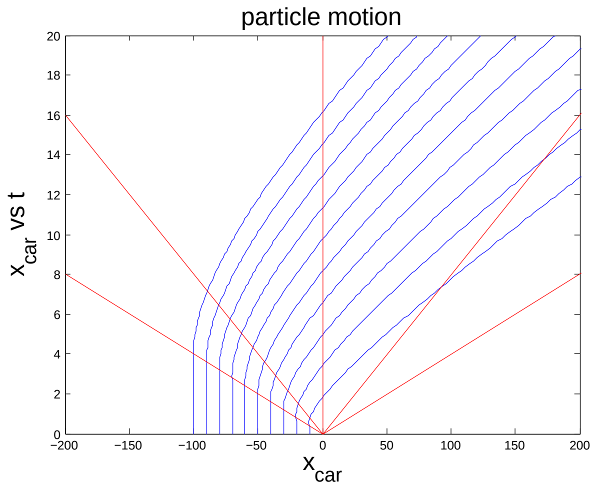

FIG. 3: Individual car trajectories.

[See Chapter 8, Section 4 of Boas.] In Figs. 3,4,and 5 there are results from integration of Eq. (15) for the traffic problem. [Numerical work was done with the Lax-Wendroff integration scheme. The shock feature at the back of the stopped car density is not easily handled analytically. The car at the front of the density obey Eq.(22). At least for a while. The numerical work was done for a periodic system. So cars leaving from the right return on the left. This makes for more interesting motion of the shock.] Look at Figs. 4 and 5. Cars peel off the front of the initial density profile in accord with your experience. The shock, density discontinuity, stands still until it get freed up by the characteristic $t = -x/v_m$. Then the shock begins to move, first slowly and then more rapidly, cars leave the front of the shock and join it from the rear. As time evolves the amplitude of the shock decreases. At long time expect a uniform density of cars driving at a constant speed set by the initial density averaged over the space available.

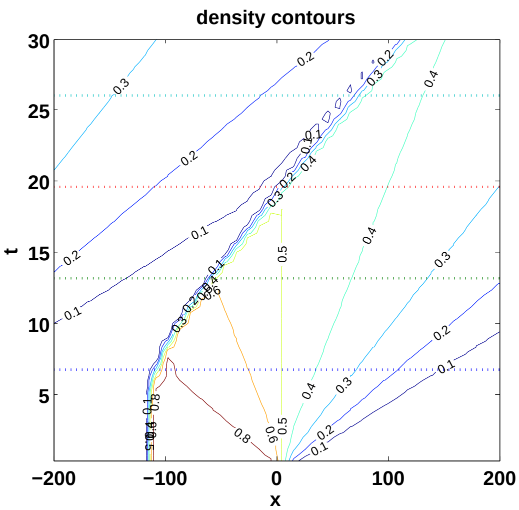

FIG. 4: Density of cars as a function of x and t.

It is the shock that is of most interest.

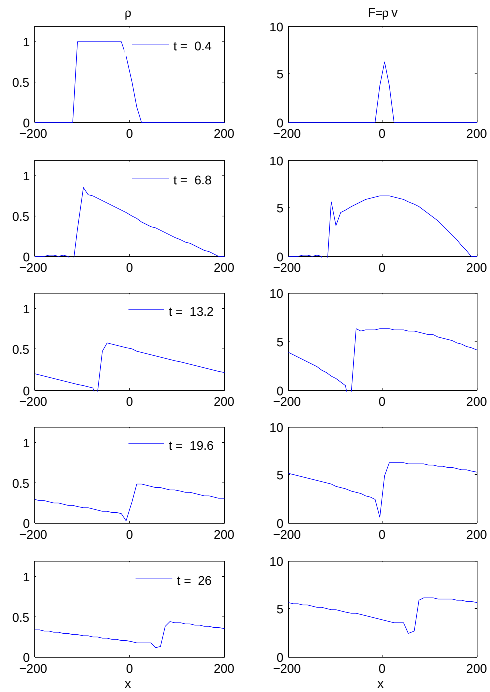

FIG. 5: Density (left column) and flux F (right column) as a function of x for 5 times. These times correspond to the dashed lines on the density profile in Fig. 4.