Physics 740: Spring 2007: 03/28/07

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Waves III: Solitary Waves and Solitons.

1. Recall. For the sine-Gordon equation, in the Taylor series approximation, Eq. (35) of Note 12, we found the one "kink" solution

$$\phi(x - vt) = \sqrt{3!}\tanh\frac{\kappa(x - vt)}{\sqrt{1-\beta^2}}, \tag{1}$$

by quadrature.

Following a lengthy manipulation of the equations for an incompressible fluid with a free surface we came to the equation for the surface displacement, ζ, when weak dispersion and nonlinearity were present,

$$\left(1 - \frac{gh_0}{v^2}\right)\zeta - \frac{3}{2h_0}\zeta^2 - \frac{h_0^2}{3}\zeta'' = 0, \tag{2}$$

the KdV equation. We wish to solve this equation for ζ. In principle this equation can be solved by quadrature just like sine-Gordon. It can be solved approximately by a variational scheme. It can also be solved by the IG method, as was done in class. In the IG method one makes an insightful guess and follows the consequences.

2. Solution to the KdV equation.

1. Further sterilize the equation by using

$$z = \frac{\eta}{h_0} = \frac{x - vt}{h_0}, \tag{3}$$

$$\nu = \frac{\zeta}{h_0}, \tag{4}$$

to find

$$\alpha_2\nu'' = \beta\nu - \alpha_1\nu^2, \tag{5}$$

where $\nu' = d\nu/dz$, $\beta = (1 - c_0^2/v^2)$, $\alpha_2 = 1/3$ and $\alpha_1 = 3/2$.

2. For the IG use

$$\nu = A\ sech^2(\kappa z), \tag{6}$$

where A and κ are constants to be found. For ν'' find

$$\nu'' = 4\kappa^2 A\mathcal{S}^2 - 2\kappa^2 A\mathcal{S}^4 \tag{7}$$

where $\mathcal{S}$ is shorthand for $sech(\kappa z)$.

3. Assemble Eq. (5)

$$4\alpha_2\kappa^2 A\mathcal{S}^2 - 2\alpha_2\kappa^2 A\mathcal{S}^4 = \beta A\mathcal{S}^2 - \alpha_1 A^2\mathcal{S}^4 \tag{8}$$

and require that the coefficient of like powers of $\mathcal{S}$ vanish (why is this a valid argument?). Find

$$4\kappa^2\alpha_2 = \beta, \tag{9}$$

$$2\kappa^2\alpha_2 = \alpha_1 A. \tag{10}$$

There are two results.

(a) The width of the soliton, κ, depends on the amplitude,

$$\kappa = \sqrt{\frac{\alpha_1}{2\alpha_2}A}. \tag{11}$$

(b) The velocity of the soliton depends on its amplitude

$$v = c_0(1 + \alpha_1 A). \tag{12}$$

When all of the pieces are put back together find

$$\zeta(x,t) = \zeta(0)sech^2\left(\kappa(\zeta(0))\frac{x - v(\zeta(0))t}{h_0}\right), \tag{13}$$

where

$$\kappa = \sqrt{\frac{\alpha_1}{2\alpha_2}\frac{\zeta(0)}{h_0}}, \tag{14}$$

$$v = c_0\left(1 + \alpha_1\frac{\zeta(0)}{h_0}\right). \tag{15}$$

There are 3 lengths in the description of this soliton, $h = h_0$, $a = \zeta(0)$ and $L = h_0\kappa^{-1}$. The solitons existence depends upon a balance of dispersion against nonlinearity. One can see what is called for qualitatively by comparing the nonlinear and dispersion terms in Eq. (2)

$$\frac{3}{2h_0}\zeta^2 \sim \frac{a^2}{h_0}, \tag{16}$$

$$\frac{h_0^2}{3}\zeta'' \sim h_0^2\frac{a}{L^2}. \tag{17}$$

When these terms to balance the geometrical features of the soliton obey

$$aL^2 \ \sim \ h_o^3. \tag{18}$$

Of course that is what the exact solution does. We have

$$\zeta(0)L^2 = \zeta(0)\frac{h_0^2}{\kappa^2} = h_0^3. \tag{19}$$

3. A Variational Principle. In the two cases for which we have exact solutions we were able to look at the problem from a pseudo classical mechanics point of view, i.e., as $T + V = E$, Eq. (37) in Note 12. Suppose at a fixed moment of time we imagine the energy of the system to be able to be represented by

$$\mathcal{E}[\nu] = \int \, dz \left[\frac{\alpha_2}{2} \left(\frac{d\nu}{dz} \right)^2 + \beta \frac{\nu^2}{2} - \alpha_1 \frac{\nu^3}{3} \right]. \tag{20}$$

The equation of motion would follow upon varying $\mathcal{E}(\nu)$ with respect to ν. Show this. When you are not able to (or do not want to) solve the resulting differential equation it is possible to carry out a numerical variation that will provide the essential structure of the solution. Variational schemes familiar from all over physics, quantum mechanics, classical fields, E and M, etc. For the case at hand as one is looking for a spatially local solution you might use a trial function

$$\nu_T = B \ exp(-\gamma^2 z^2/2), \tag{21}$$

that has two variational parameters, B that sets the amplitude of the solution and γ that sets the spatial extent of the solution. Calculate

$$\mathcal{E}_T = \int \, dz \left[\frac{\alpha_2}{2} \left(\frac{d\nu_T}{dz} \right)^2 + \beta \frac{\nu_T^2}{2} - \alpha_1 \frac{\nu_T^3}{3} \right]. \tag{22}$$

vary the trial solution with respect to its parameters, i.e., solve

$$\frac{\partial \mathcal{E}_T}{\partial B} = 0 \ \ , \tag{23}$$

$$\frac{\partial \mathcal{E}_T}{\partial \gamma} = 0. \tag{24}$$