Physics 740: Spring 2006: 02/14/07

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Vorticity and vortices. Vorticity occurs in many scenarios in fluid dynamics. The case of an ideal fluid, i.e., no viscosity, is special and set of scenarios is much more limited. However, problems involving ideal fluid vortices map onto a number of other problems of considerable importance (most of which are two dimensional): the two dimensional Coulomb gas, the two dimensional X-Y model, D=2 superconductivity, D=2 superfluidity, the roughening transition, certain liquid crystals, etc. While the details differ all of these cases involve vortices that have a well defined strength, circulation, but two signs (directions of motion of the fluid around the vortex core. [Maybe the most glamorous example is the two dimensional superfluid for which the circulation about a vortex core is quantized (essentially as Bohr would do it). So in superfluids (superconductors) there are quanta of circulation. They come with two *signs* as the circulation about the vortex core can be clockwise or counter-clockwise. Hence, the association with the two dimensional Coulomb gas.]

1. Velocity field and energy of a vortex.

(a) Velocity field. Consider a vortex core strung from top to bottom of a finite cylindrical container. Assume the velocity field of the fluid is axially symmetric, i.e., **v**=v(r)$\mathbf{e}_r$. Then

$$\oint \mathbf{v} \cdot d\mathbf{r} = \oint v(r) dr = 2\pi r v(r) = \kappa, \tag{1}$$

$$v(r) = \frac{\kappa}{2\pi r}, \quad r \geq a, \tag{2}$$

where κ is the circulation of the vortex and a is the radius of the vortex core. There are two possible directions for he fluid flow around a vortex core. Use some sign convention and a right hand rule. Then, each circulation will have a well defined sign. For simplicity we will consider the case of vortices in a superfluid for which the numerical values of the circulation are quantized in units of $\hbar/m$, $\kappa = \pm n\hbar/m$. [It is found empirically that almost always $n = 1$.]

Aside. You might understand Bohr quantization by the argument of deBroglie, the wave describing a particle must be periodic over the orbit of the particle. Since $p = h/\lambda$ *Bohr follows. A superfluid is a macroscopic collection of particles described by a single quantum mechanical wavefunction (that's why the collection of particles moves coherently) that must*

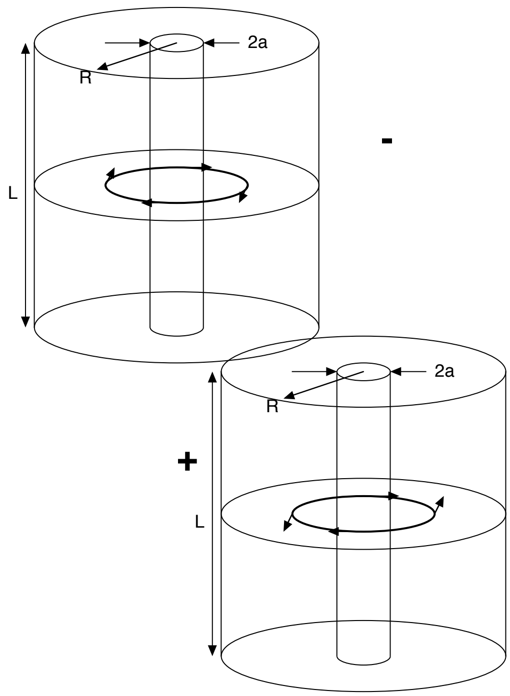

FIG. 1: A vortex at the center of a cylinder of radius R and length L. The vortex core has radius a and $v(r) \sim 1/r$.

be periodic over a closed path in the fluid. In the case of a superfluid the quantity that must be periodic corresponds to a flow of the fluid. (Maybe the same is true of an electron flowing around a proton in a H atom. How would you know?)
End aside.

(b) Energy. The energy in a vortex is made up of two parts, 1. the energy to form the vortex core and 2. the energy in the flow of the fluid.

1. Core. This energy is a constant and the same for all vortex cores. There are theories of this energy, it is of courese proportional to the length of the core, denote it as ϵ_0. As it does not depend on where the vortex is or on the position of a vortex relative to other vortices it is unimportant. Rather like the self energy of an electron.

2. Kinetic energy. The important energy of a vortex is the kinetic energy in the fluid flow around the core. This is

$$\mathcal{E} = \int_a^R KE(r)\ 2\pi r dr = \int_a^R \frac{\rho}{2} v(r)^2\ 2\pi r dr = \frac{\rho\kappa^2}{4\pi}\ ln\frac{R}{a}, \tag{3}$$

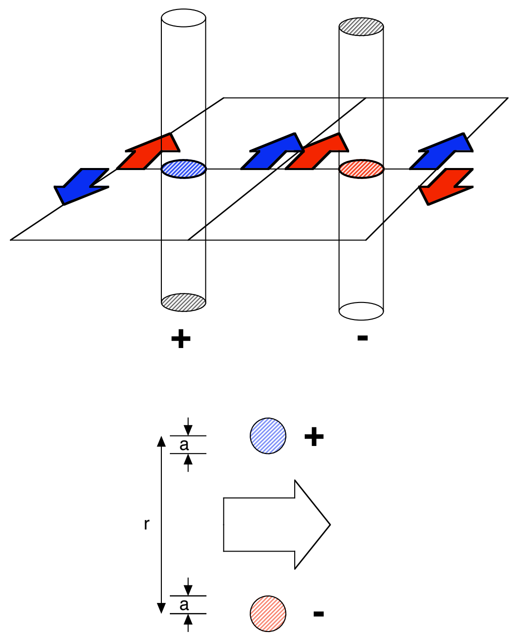

FIG. 2: The flow field of the + vortex (blue) adds to that of the − vortex (red) in the space between them. Beyond the two cores the flow fields tend to cancel so that the important kinetic energy is in a volume of space of order $r^2 L$.

where R is a practical upper limit to the region of space associated with the vortex (possibly the radius of the cylinder). Note that $\mathcal{E}$ is an energy per unit length as it should be. The total kinetic energy in the fluid flow is $L\mathcal{E}$.

Aside. The result here is the same as that for the electric field of a uniform line charge, $\mathbf{E}(r) \propto (\lambda/r)\mathbf{e}_r$*, with* λ *the charge per unit length. The energy in the electric field is proportional to the integral of* $\mathbf{E}(r) \cdot \mathbf{E}(r)$ *over volume. When the field lines from* λ *terminate on a distant surface you get the result in Eq. (3).*
End aside.

2. Velocity field and energy of a vortex pair. Consider a pair of vortices of equal and opposite vorticity separated by distance r. The energy of the pair is $2\epsilon_0$, for the two cores, plus the kinetic energy of the flow field. Between the two vortices the flow fields add. Outside of the two vortices they tend to cancel one another. (Just like the electric field of two line charges that have equal and opposite λ, i.e., a cylindrical dipole.) See Fig. 2.

The energy of a pair of vortices is approximately

$$\mathcal{E}_2 = \int_{-\frac{r}{2}+a}^{\frac{r}{2}-a} \frac{\rho}{2}(v_+(r) + v_-(r))^2 \; 2\pi r dr \approx -\rho\kappa^2 \; ln\frac{r}{a}, \tag{4}$$

A careful calculation yields

$$\mathcal{E}_2 = -\frac{\rho}{4\pi}\kappa^2 \; ln\frac{r}{a}, \tag{5}$$

and for a system of N vortices, varying circulation,

$$\mathcal{E}_N = -\frac{\rho}{4\pi} \; \sum_{i=1}^{N}\sum_{j=1}^{N} \kappa_i\kappa_j \; ln\frac{r_{ij}}{a_{ij}}, \tag{6}$$

where $a_{ij} = a_i + a_j$.

3. Equations of motion of vortices. The equation of motion is a vortex is a realization of the consequences of the Kelvin *circulation theorem*. The vorticity moves with the fluid. Thus

$$\frac{dx_i}{dt} = \sum [x \; velocity \; at \; (x_i, y_i) \; from \; vortices \; j = 1 \cdots N \; (except \; i)], \tag{7}$$

$$\frac{dy_i}{dt} = \sum [y \; velocity \; at \; (x_i, y_i) \; from \; vortices \; j = 1 \cdots N \; (except \; i)], \tag{8}$$

or

$$\frac{dx_i}{dt} = -\frac{1}{2\pi}\sum_j{}' \kappa_j \frac{(y_i - y_j)}{r_{ij}^2}, \tag{9}$$

$$\frac{dy_i}{dt} = +\frac{1}{2\pi}\sum_j{}' \kappa_j \frac{(x_i - x_j)}{r_{ij}^2}. \tag{10}$$