P740.HW4.tex

Due 03/05/07

1. n-stars: Background. When a massive star (much larger than the sun) runs out of nuclear fuel it no longer has the thermal means to support itself against collapse due to gravity. It goes through a complex cycle ending in an explosion which throws much of its mass out into space and often leaves behind a cinder of material having a mass of order that of the Sun. This material, a collection of N electrons and N protons (p, e), can be regarded as being at $T = 0$K. Lacking thermal support this material is pulled inward by its own gravity. The electrons find themselves confined to a space of volume $v = V/N$ (roughly, the Pauli principle forces each electron into its share of the cinder's volume V) where its energy of localization, $\hbar^2/(m_e v^{2/3})$, becomes so great that the electrons are absorbed into the protons, $e + p \rightarrow n$ (and neutrinos). The cinder, made of neutrons, is a chunk of neutron matter. When the neutron matter is crushed to the density of nuclear material it is able to support itself against further collapse using the Pauli pressure of the neutrons. [These are cold fermi systems so Pauli is everywhere.]

1. Take the cinder to have the mass of the Sun and the number density of nuclear material to be $n_N = a_N^{-3}$, where $a_N = 1$ fermi. Estimate the radius of the cinder, a spherical object. State your answer in McCarrans. [The McCarran is a unit of length determined by a circle approximation to McCarran Blvd (Reno, NV), $C = 2\pi R_{McC}$, the length R_{McC} is one McCarran.]

2. The cinder, a neutron star, was created by central forces (gravity) so its final angular momentum is the same as its initial angular momentum. Take the initial angular momentum/size of the cider to be that of the Sun. How fast is the cinder rotating? A rotating neutron star is a pulasr. Pulsar rotation rates vary widely but a period of 1 sec is not unusual, a millisecond is rare. Could the Sun rotate with a period of 1 sec?

3. In principle the density of the neutron star obeys the equation of hydrostatic equilibrium

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(\frac{r^2}{\rho}\frac{\partial P}{\partial r}\right) = -4\pi G\rho, \quad 0 \leq r \leq R, \quad P(R) = \rho(R) = 0, \tag{1}$$

where the pressure-density equation of state of $T = 0\text{K}$ neutrons is

$$P \sim \frac{N}{V}\frac{\hbar^2}{m_n}\left(\frac{N}{V}\right)^{\frac{2}{3}}, \tag{2}$$

with $\rho = m_n(N/V)$. [This pressure is the Pauli pressure of the neutrons.] Use $x = r/R$ and $y = \rho/(M/R^3)$ to write Eq. (1) in dimensionless form. The resulting differential equation has a single constant involving physical variable, i.e., M, G, R, $\hbar$, $\cdots$, that is the ratio of two pressures, a gravitational pressure, P_G, and a Pauli pressure, P_P. Find these two pressures. Estimate their size.

4. From Eq. (2) you can find $\partial P/\partial\rho$. Imagine that the material being discussed in the note **P740.1.tex** was neutron matter. Estimate the velocity of sound in neutron matter. The mechanical vibrations of a material, say the breathing mode of a star, have frequencies of order the time for sound to cross the material. What sort of vibrational frequencies are associated with a neutron star? Could the Sun vibrate at these frequencies?

2. Something about vortices. In the note **P740.8.tex**, Fig. 1 (a hard copy was handed out in class) the equations solved to find the motion pictured were Eqs. (9) and (10) in note **P740.7.tex** with the factor 2π absorbed into the definition of κ.

1. For the lower two figures find the equation for $\mathbf{x}_1(t)$and $\mathbf{x_2}(t)$ that solves Eqs. (9) and (10). Do this for general values of K_1 and K_2 but for $\mathbf{x}_1(0) = (0, 1)$and $\mathbf{x_2}(0) = (0, -1)$; do it by inspecting the figures and making a plausible guess as to what you see.

2. If you wanted to integrate the differential equations, (9) and (10), that describe the motion of the vortices you would begin by writing out the 4 equations called for, i.e.,

$$\dot{x}_1 = -K_2\frac{y_1 - y_2}{r_{12}^2} = -K_2 Y, \tag{3}$$

$$\dot{y}_1 = K_2\frac{x_1 - x_2}{r_{12}^2} = K_2 X, \tag{4}$$

$$\dot{x}_2 = -K_1\frac{y_2 - y_1}{r_{12}^2} = K_1 Y, \tag{5}$$

$$\dot{y}_2 = K_1\frac{x_2 - x_1}{r_{12}^2} = -K_1 X, \tag{6}$$

four coupled, first order ODEs. The simplest integration scheme is the Euler method,

(a) you want to learn x_1, x_2, y_1, y_2 at M moments of time $t_m = (m-1)\Delta t$, $m = 1 \cdots M$, between the initial time $t_1 = 0$ and the final time $t_M = (M-1)\Delta t = t_{max}$. You get to choose Δt and M.

(b) replace time derivatives by (example)

$$\dot{x}_1 = \frac{x_1(t_{m+1}) - x_1(t_m)}{\Delta t} \tag{7}$$

and write

$$x_1(t_{m+1}) = x_1(t_m) - \Delta t \; K_2 \; Y(t_m), \tag{8}$$

$$y_1(t_{m+1}) = y_1(t_m) + \Delta t \; K_2 \; X(t_m), \tag{9}$$

$$x_2(t_{m+1}) = x_2(t_m) + \Delta t \; K_1 \; Y(t_m), \tag{10}$$

$$y_2(t_{m+1}) = y_2(t_m) - \Delta t \; K_1 \; X(t_m). \tag{11}$$

This is a system of equations that can be iterated, i.e., at $m = 1$ ($t_1 = 0$) you use $x_1(t_1 = 0), x_2(t_1 = 0), y_1(t_1 = 0), y_2(t_1 = 0)$ to evaluate the X and Y and anything else on the RHS and learn $x_1(t_2), x_2(t_2), y_1(t_2), y_2(t_2)$ using Eqs. (8)-(11). With $x_1(t_2), x_2(t_2), y_1(t_2), y_2(t_2)$ you repeat the process to learn $x_1(t_3), x_2(t_3), y_1(t_3), y_2(t_3)$, etc.

(c) a scheme like this is begging to be put into a **for** loop, e.g.

```
for ii=2:NT
Dx12=x1(ii-1)-x2(ii-1);
Dy12=y1(ii-1)-y2(ii-1);
r2=(Dx12*Dx12+Dy12*Dy12);
Fx=Dy12/r2;
Fy=Dx12/r2;
x1(ii)=x1(ii-1)-dt*K2*Fx;
x2(ii)=x2(ii-1)+dt*K1*Fx;
y1(ii)=y1(ii-1)+dt*K2*Fy;
y2(ii)=y2(ii-1)-dt*K1*Fy;
end
```

Input to this loop are the 6 numbers K_1, K_2, $\mathbf{x}_1(0)$ and $\mathbf{x}_2(0)$ and dt.

(d) Carry through this scheme for $(K_1, K_2) = (1, -1)$, $(1, 1)$, $(1, 2)$, $(1, 3)$, $(1, 4)$, $(1, -3)$.

(e) The choice you make for Δt should be such that $x(t_{m+1})$ does not differ greatly from $x(t_m)$. So you try some values. See how it goes. Eventually you learn that there are more cool method than the Euler method. These are adaptive time step methods, where the iteration scheme keeps adjusting Δt as you go along so that you don't take too big a step, don't take such itty-bitty steps that you get nowhere. Matlab, Mathematica, ... have various ODE solvers that have these features. [At least in Matlab they are not completely transparent. It never hurts to try Euler to get started and go to **ode45**, etc. for refinement if needed.]

3. More about a vortex. Solve Problem 1, page 21 in the text, for the case that the cylindrical velocity field is that of a quantized vortex

$$\mathbf{v}(r) = \frac{\hbar}{m}\frac{1}{r}\mathbf{e}_\theta, \quad r \geq a. \tag{12}$$

Find $z(r)$ and estimate the depth of the *dimple* caused by the vortex for the case of superfluid ^{4}He, i.e., $m = 4$ amu. For the core radius use σ for ^{4}He in the Table at the top of **P740.HW2.tex**.