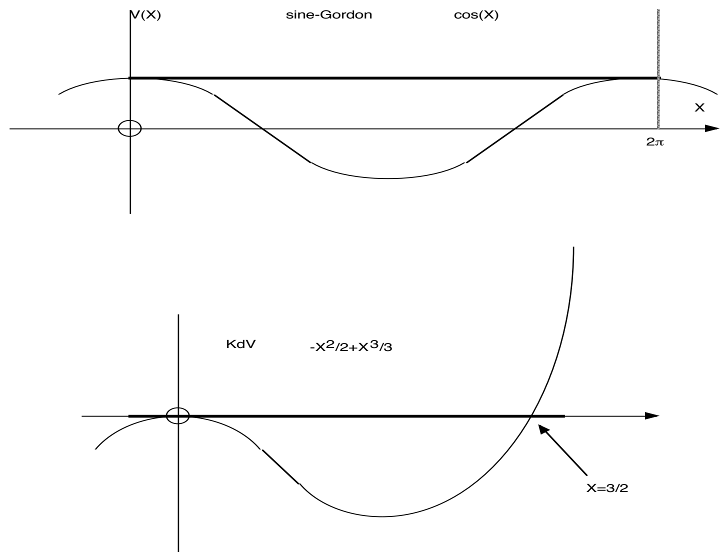

FIG. 1: sine Gordon and KdV potentials. The energy is shown as a heavy black line.

P740.HW6.sol.tex

1. sine-Gordon by quadrature. The equation to be solved is

$$\ddot{X} = \sin X. \tag{1}$$

Multiply by $\dot{X}$ and integrate to find

$$\frac{\dot{X}^2}{2} = E - \cos X = E - V(X). \tag{2}$$

See Fig. 1. A solution for $V(X) = \cos X$ that starts at $X = 0$ and evolves to $X = 2\pi$ can be found for the choice $E = 1$. For this case

$$\frac{\dot{X}}{2} = \sin \frac{X}{2}. \tag{3}$$

Use $x = X/2$ to sterilize further

$$\dot{x} = \sin x \rightarrow \frac{dx}{\sin x} = dz. \tag{4}$$

[Warning. The most sterile possible form of all equations is desirable when using algebra programs like Mathematica, Maple, ... These programs are so smart that they see everything that is not tied down as a complex number or worse. Give them nothing to think about but the most rudimentary forms.] From Dwight 432.10 the solution to this equation is

$$\log\left(tan\frac{x}{2}\right) = z. \tag{5}$$

Unscrambling

$$X = 4tan^{-1}(\ e^z\ ). \tag{6}$$

From the Note 12, $z = k_0(x - vt)/\sqrt{1-\beta^2}$, $\beta = v/c_0$ and $k_0 = \sqrt{A}/c_0$. $X(z \to -\infty) = 0$, $X(z = 0) = \pi$ and $X(z \to +\infty) = 2\pi$.

2. KdV by quadrature. The equation to be solved is, Eq. (5) of Note 13,

$$\alpha_2\ddot{\nu} = \beta\nu - \alpha_1\nu^2. \tag{7}$$

Sterilize. Divide by β. Define $\tau = dz\sqrt{\beta/\alpha_2}$. Scale by $\lambda = \beta/\alpha_1$, i.e., $\nu = \lambda X$. Find

$$\ddot{X} = X - X^2. \tag{8}$$

Multiply by $\dot{X}$ and integrate to find

$$\frac{\dot{X}^2}{2} = E + \frac{X^2}{2} - \frac{X^3}{3} = E - V(X). \tag{9}$$

See Fig. 1. A solution for $V(X)$ that starts at $X = 0$ and evolves back to X can be found for the choice $E = 0$. For this case

$$\dot{X} = \sqrt{X^2 - \frac{2X^3}{3}}. \tag{10}$$

From Dwight 192.11 the solution to this equation is

$$tanh^{-1}\sqrt{1 - 2X/3} = -\frac{\tau}{2}. \tag{11}$$

Unscrambling

$$X = \frac{3}{2}\ sech^2\left(\frac{\tau}{2}\right). \tag{12}$$

Further unscramble by going through the variable changes. Compare to Eq. (13) in note 13.

3. KdV by variational principle. [Note the quantity in the discussion on page 3 of Note 13 is $KE - V$, the Lagrangian density, e.g., FW chapter 7.] Why not do this in the sterilized form from Eq. (9) above

$$\mathcal{E} = \int d\tau \, \left(\frac{\dot{X}^2}{2} + \frac{X^2}{2} - \frac{X^3}{3} \right) = \frac{< \dot{X}^2 >}{2} + \frac{< X^2 >}{2} - \frac{< X^3 >}{3}. \tag{13}$$

With $X_T = Bexp(-\gamma^2 \tau^2/2)$ find

$$< \dot{X}^2 > = \frac{\sqrt{\pi}\gamma B^2}{2}, \tag{14}$$

$$< X^2 > = \frac{\sqrt{\pi} B^2}{\gamma}, \tag{15}$$

$$< X^3 > = \frac{\sqrt{2\pi} B^3}{\sqrt{3}\gamma}, \tag{16}$$

and

$$\mathcal{E} = \frac{\sqrt{\pi}}{2} \left(\frac{\gamma B^2}{2} + \frac{B^2}{\gamma} - b\frac{B^3}{\gamma} \right), \tag{17}$$

where $b = (2/3)^{3/2}$. Find $d\mathcal{E}/dB$ and $d\mathcal{E}/d\gamma$ with the results

$$B = \sqrt{\frac{3}{2}} \left(1 + \frac{\gamma^2}{2} \right), \tag{18}$$

$$\gamma^2 = 2(1 - bB). \tag{19}$$

Combine, $\gamma^2 = 2/5$ and $B = (6/5)\sqrt{3/2}$, just numbers. Work back through the definitions and find

$$\zeta = Ch_0 \beta \; exp \; - \left(\frac{3}{5} \beta \left\{ \frac{x - vt}{h_0} \right\}^2 \right). \tag{20}$$

4. Soliton Collisions. Consider the pendulum model of the sine-Gordon system. A soliton can be defined by its helicity. Use a right hand rule. For a **solition**: when your thumb is in the direction of motion of the soliton the pendulums twist in the sense of your fingers. For an anti-soliton they twist in the opposite sense. See Fig. 2. Upper left: soliton going from left to right. The heavy black pendulum will go through the sequence black, red, blue $\cdots$ as the soliton passes by. The scattering rules come from examining these figures.

1. Two solitons approaching one another put incompatible demands on the black pendulum, the solitions repel.

2. Two anti-solitons approaching one another put incompatible demands on the black pendulum, the anti-solitions repel.

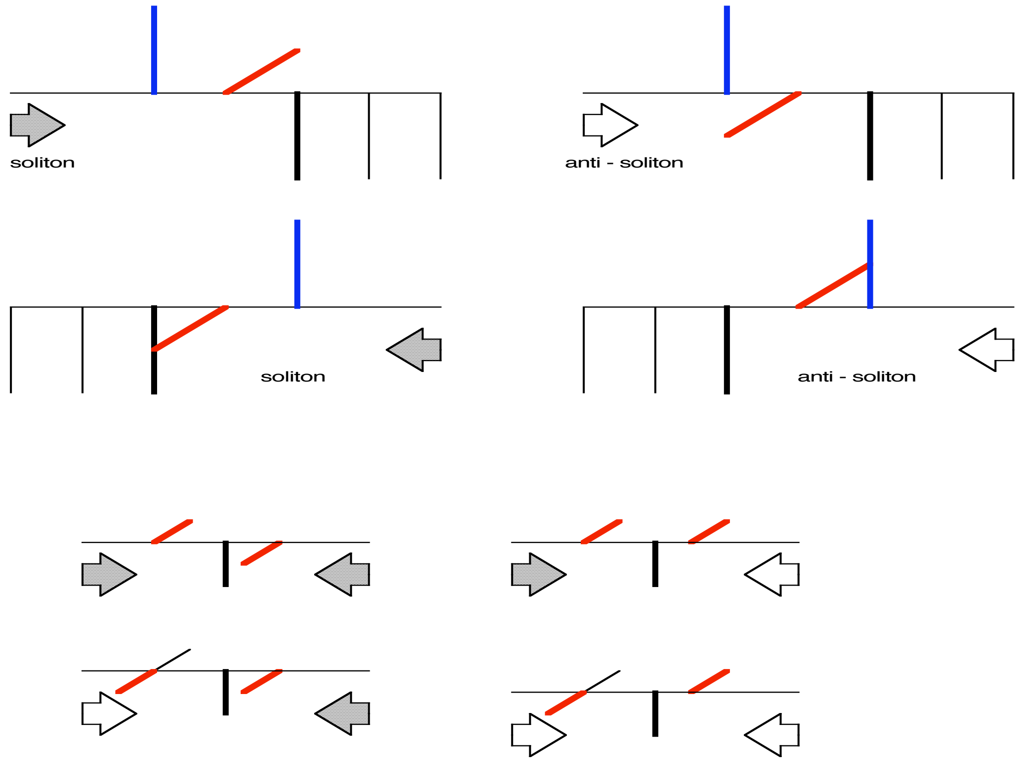

FIG. 2: Soliton Collisions.

3. A soliton and anti soliton approaching one another put compatible demands on the black pendulum, the solitions pass by one another.