Physics 701 Fall 2006, 2nd Assignment

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1 Show that $\hat{n} \times (\vec{H}_2 - \vec{H}_1) = \vec{K}$

$\vec{K}$ is surface current density and $\hat{n}$ is unit vector, normal to the surface and out of the medium. Before solving, let's assume that $H_2 > H_1$
All vectors and the boundary are represented on the Figure 1 (additional page of the document).

Basically, Maxwell equation $\vec{\nabla} \times \vec{H} = \vec{J}$ states that curl of the magnetic field is caused by the current. So we study the current at the boundary. It wouldn't be wise to use differential form of Maxwell equation, because we are dealing with boundary. Magnetic fields are different in both media. So we'd rather use integral form of the Maxwell equation and we'll take path integral through the points (0,0), (dx,0), (dx,dy), (0,dy), (0,0). So the equation would be:
$\oint_C \vec{H} \cdot d\vec{l} = \int_S (\vec{\nabla} \times \vec{H}) \cdot d\vec{A}$
We can write left side as: $H_2 dx + H_x dy - H_1 dx - H_x dy = H_2 dx - H_1 dx$ We don't really care about H_x because it zeroes out anyway. Right side of the equation is basically the current per surface area. As we are considering only the boundary, we can write the right side: $K \cdot dx$ because $\vec{K} \parallel \vec{dx}$
So alltogether we have: $H_2 - H_2 = K$ But we want to show these as vectors. Let's take vector $\hat{n} = -\hat{y}$ and let's write the left side of the equation as:
$\hat{n} \times (H_2 \hat{x} - H_1 \hat{x}) = \hat{z}(H_2 - H_1)$ and as $H_2 - H_1 = K$ from Maxwell equation, so we can write the result as:
$\hat{n} \times (\vec{H}_2 - \vec{H}_1) = \vec{K}$

2 Prove that $\Delta \vec{A} = -\mu \vec{J}$

We are using Maxwell equation $\vec{\nabla} \times \vec{B} = \mu \vec{J} + \epsilon_0 \frac{\partial \vec{E}}{\partial t}$. As we are dealing only with steady currents, the term $\epsilon_0 \frac{\partial \vec{E}}{\partial t} = 0$

Magnetic induction vector can be written using magnetic vector potential:
$\vec{B} = \vec{\nabla} \times \vec{A}$ Using results from the last task of this assignment, we can write the left side of the Maxwell equation as follows:
$\vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{A}) - \Delta\vec{A}$
And we know that $\vec{\nabla} \cdot \vec{A} = 0$, so the left side of the equation will be:
$\vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = -\Delta\vec{A}$
And considering fact that we are dealing with steady currents, we can write:

$\Delta\vec{A} = \mu\vec{J}$

3 Show that

a)$\vec{\nabla} \cdot [\vec{r} \cdot f(r)] = 3f(r) + r \cdot \frac{df(r)}{dr}$

The solution:
$\vec{\nabla} \cdot [\vec{r} \cdot f(r)] = \frac{\partial}{\partial x_1}(x_1 f(r)) + \frac{\partial}{\partial x_2}(x_2 f(r)) + \frac{\partial}{\partial x_3}(x_3 f(r)) =$
$= f(r) + x_1 \frac{\partial}{\partial x_1} f(r) + f(r) + x_2 \frac{\partial}{\partial x_2} f(r) + f(r) + x_3 \frac{\partial}{\partial x_3} f(r) =$
$= 3f(r) + x_1 \frac{\partial f}{\partial r}\frac{\partial r}{\partial x_1} + x_2 \frac{\partial f}{\partial r}\frac{\partial r}{\partial x_2} + x_3 \frac{\partial f}{\partial r}\frac{\partial r}{\partial x_3} = 3f(r) + \frac{df}{dr}(x_i \frac{\partial r}{\partial x_i}) =$
$= 3f(r) + \frac{df}{dr}(\frac{x_1^2}{r} + \frac{x_2^2}{r} + \frac{x_3^2}{r}) = \mathbf{3f(r) + r\frac{df}{dr}}$

b)$\vec{\nabla} \cdot (\vec{r} \cdot r^{n-1})$

At first we'd we can expand the contents in brackets and the result would be:
$\vec{\nabla} \cdot [(x_1,\ x_2,\ x_3) \cdot (x_1^2 + x_2^2 + x_3^3)^{\frac{n-1}{2}}]$

So now we have divergence of vector with all components explicitly written down. We can take partial derivatives from each component now:
$\frac{\partial}{\partial x_1}(x_1 \cdot (x_1^2 + x_2^2 + x_3^3)^{\frac{n-1}{2}}) + \frac{\partial}{\partial x_2}(x_2 \cdot (x_1^2 + x_2^2 + x_3^3)^{\frac{n-1}{2}}) + \frac{\partial}{\partial x_3}(x_3 \cdot (x_1^2 + x_2^2 + x_3^3)^{\frac{n-1}{2}}) =$
$= 3r^{n-1} + \frac{\partial}{\partial x_1}((x_1^2 + x_2^2 + x_3^3)^{\frac{n-1}{2}}) \cdot x_1 + \frac{\partial}{\partial x_2}((x_1^2 + x_2^2 + x_3^3)^{\frac{n-1}{2}}) \cdot x_2 + \frac{\partial}{\partial x_3}((x_1^2 + x_2^2 + x_3^3)^{\frac{n-1}{2}}) \cdot x_3$

Now we have to take partial derivatives of those componenents and the result is:
$3r^{n-1} + \frac{n-1}{2} \cdot r^{\frac{n-3}{2}} \cdot 2x_1 \cdot x_1 + \frac{n-1}{2} \cdot r^{\frac{n-3}{2}} \cdot 2x_2 \cdot x_2 + \frac{n-1}{2} \cdot r^{\frac{n-3}{2}} \cdot 2x_3 \cdot x_3 =$
$= 3r^{n-1} + (n-1) \cdot r^{\frac{n-3}{2}}(x_1^2 + x_2^2 + x_3^2)] = 3r^{n-1} + (n-1) \cdot r^{\frac{n-3}{2}} r^2 = \mathbf{r^{n-1}[3 + (n-1) \cdot r^{-2}}]$

4 Vector identities:

1. Simplify $[\vec{A} \times (\vec{B} \times \vec{C})]_i$and compare to $[(\vec{A} \times \vec{B}) \times \vec{C}]_i$

(a) $[\vec{A}\times(\vec{B}\times\vec{C})]_i = \epsilon_{ijk}A_j\epsilon_{klm}B_lC_m = \epsilon_{kji}\epsilon_{klm}A_jB_lC_m =$
$= (\delta_{il}\delta_{jm} - \delta_{jm}\delta_{jl})A_jB_lC_m = A_jB_iC_j - A_jB_jC_i = (A\cdot C)B_i - (A\cdot B)C_i$

(b) $[(\vec{A}\times\vec{B})\times\vec{C}]_i = \epsilon_{ijk}\epsilon_{jlm}A_lB_mC_k = -\epsilon_{jik}\epsilon_{jlm}A_lB_mC_k =$
And as we did previously, we can write result:
$= (A\cdot C)B_i - (B\cdot C)A_i$
As we can see, the results are not equal and the second identity can be written as:
$[-\vec{C}\times(\vec{A}\times\vec{B})]$

2. Express $(\vec{A}\times\vec{B})\cdot(\vec{C}\times\vec{D})$ using only scalar products of the vectors:
$(\vec{A}\times\vec{B})\cdot(\vec{C}\times\vec{D}) = (\vec{A}\times\vec{B})_i\cdot(\vec{C}\times\vec{D})_i = \epsilon_{ijk}\epsilon_{ilm}A_jB_kC_lD_m =$
$= (\delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl})A_jB_kC_lD_m =$
$= A_jB_kC_jD_k - A_jB_kC_kD_j = (A\cdot C)(B\cdot D) - (A\cdot D)(B\cdot C)$

3. $[\vec{\nabla}\times(\phi\vec{A})]_i = \epsilon_{ijk}\frac{\partial\phi A_k}{\partial x_j} = \sum_i\sum_k \epsilon_{ijk}(A_k\frac{\partial\phi}{\partial x_j} + \phi\frac{\partial A_k}{\partial x_j})$
We can write this down explicitly for the case when i=1:
$= A_3\frac{\partial\phi}{\partial x_2} + \phi\frac{\partial A_3}{\partial x_2} - A_2\frac{\partial\phi}{\partial x_3} - \phi\frac{\partial A_2}{\partial x_3}$ and it's not hard to see that general result is quite simple:
$[\vec{\nabla}\times(\phi\vec{A})] = (\vec{\nabla}\phi)\times\vec{A} + \phi(\vec{\nabla}\times\vec{A})$

4. $[\vec{\nabla}(\vec{A}\cdot\vec{B})]_i = (\vec{\nabla}\cdot\vec{B})A_i - [(\vec{\nabla}\times\vec{A})\times\vec{B}]_i$ This relation comes from 1. identity written above. So we can write out second term in the sum and the result is:
$[\vec{\nabla}(\vec{A}\cdot\vec{B})]_i = (\vec{\nabla}\cdot\vec{B})A_i - (\vec{\nabla}\vec{B})A_i + \nabla_i(\vec{A}\cdot\vec{B}) = \nabla_i(\vec{A}\cdot\vec{B})$

5. $[\vec{\nabla}\times(\vec{A}\times\vec{B})]_i = \epsilon_{ijk}\nabla_j\epsilon_{klm}A_lB_m = \epsilon_{kij}\epsilon_{klm}\nabla_jA_lB_m = \nabla_jA_iB_j - \nabla_jA_jB_i$
(Solved in 1st identity). And there still is the restriction that $i \neq j$

6. $[\vec{\nabla}\times(\vec{\nabla}\times\vec{A})]_i = \nabla_j\nabla_iA_j - \nabla_j\nabla_jA_i$ (according to previous identity...)
But we can also prove it by using Levi Civita notation:
$[\vec{\nabla}\times(\vec{\nabla}\times\vec{A})]_i = \epsilon_{ijk}\nabla_j\epsilon_{klm}\frac{\partial A_m}{\partial x_l} = \epsilon_{kij}\epsilon_{klm}\frac{\partial}{\partial x_j}\frac{\partial A_m}{\partial x_l} = (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})\frac{\partial}{\partial x_j}\frac{\partial A_m}{\partial x_l} =$
$= \frac{\partial}{\partial x_j}\frac{\partial A_j}{\partial x_i} - \frac{\partial}{\partial x_j}\frac{\partial A_i}{\partial x_j} = \nabla_j\nabla_iA_j - \nabla_j\nabla_jA_i$

5 Successive applications of ∇ : Evaluate

1. $\vec{\nabla}\cdot\vec{\nabla}\phi = \vec{\nabla}\cdot(\hat{e}_i\frac{x_i}{r}\frac{d\phi}{dr}) = \vec{\nabla}\cdot(\hat{r}\frac{d\phi}{dr}) = \frac{d\phi}{dr}\vec{\nabla}\hat{r} + \hat{r}\vec{\nabla}\frac{d\phi}{dr} = \frac{2}{r}\frac{d\phi}{dr} + \frac{d^2\phi}{dr^2}$

2. $\vec{\nabla}\times\vec{\nabla}\phi$ What we have here is curl of a gradient. And it is always zero. It is quite easy to see from the expression: $\vec{\nabla}\times(\hat{r}\frac{d\phi(r)}{dr})$ and if we solved the equation we would have to take derivatives with respect to $\theta$and Φbut function $\phi(r)$ is not explicitly function of those coordinates.

3. Let's solve the last problem (e) now:
$\vec{\nabla} \times (\vec{\nabla} \times \vec{V}) = \vec{\nabla} \times \left[\left(\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z} \right), \left(\frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial x} \right), \left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y} \right) \right] =$
$= \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z} & \frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial x} & \frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y} \end{vmatrix} = \hat{x} \left[\frac{\partial}{\partial y} \left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y} \right) - \frac{\partial}{\partial z} \left(\frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial x} \right) \right] +$
$+ \hat{y} \left[\frac{\partial}{\partial z} \left(\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z} \right) - \frac{\partial}{\partial x} \left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y} \right) \right] + \hat{z} \left[\frac{\partial}{\partial x} \left(\frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial x} \right) - \frac{\partial}{\partial y} \left(\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z} \right) \right] =$
$= \hat{x} \left(\frac{\partial}{\partial y} \frac{\partial}{\partial x} V_y + \frac{\partial}{\partial z} \frac{\partial}{\partial x} V_z + \frac{\partial^2}{\partial x^2} V_x - \Delta V_x \right) + \hat{y}(...) + \hat{z}(...)$ We can write ΔV_x because $\frac{\partial^2}{\partial x^2} V_x$ was also added to the equation. Now we can simplify this relatively long and ugly result:
$... = \hat{x} \left[\frac{\partial}{\partial x} \left(\vec{\nabla} \vec{V} \right) - \Delta V_x \right] + \hat{y} \left[\frac{\partial}{\partial y} \left(\vec{\nabla} \vec{V} \right) - \Delta V_y \right] + \hat{z} \left[\frac{\partial}{\partial z} \left(\vec{\nabla} \vec{V} \right) - \Delta V_z \right] =$
$\vec{\nabla}(\vec{\nabla} \cdot \vec{V}) - \Delta \vec{V}$

4. Now the problem (c): $\vec{\nabla} \vec{\nabla} \vec{V} = (\vec{\nabla} \vec{\nabla}) \vec{V} + \vec{\nabla}(\vec{\nabla} \vec{V}) = \Delta \vec{V} + \vec{\nabla}(\vec{\nabla} \vec{V}) =$
$= \vec{\nabla} \times (\vec{\nabla} \times \vec{V}) + 2\Delta \vec{V} = 2\vec{\nabla}(\vec{\nabla} \vec{V}) - \vec{\nabla} \times (\vec{\nabla} \times \vec{V})$ Which ever of those results are good to use for a problem...

5. $\vec{\nabla} \cdot \vec{\nabla} \times \vec{V} = ...$ Only way we can calculate this relation is when we first calculate cross product of $\vec{\nabla} \times \vec{V}$ and then take divergence of the result. We cannot multiply nabla vectors at first because then we couldn't take cross product (which is defined as product of two vectors). And as we see, we take divergence of curl and it is always 0. Brief proof of it:
$\vec{\nabla} \cdot \left[\left(\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z} \right), \left(\frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial x} \right), \left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y} \right) \right] = \frac{\partial}{\partial x} \left(\frac{\partial V_z}{\partial y} \right) - \frac{\partial}{\partial x} \left(\frac{\partial V_y}{\partial z} \right) +$
$\frac{\partial}{\partial y} \left(\frac{\partial V_x}{\partial z} \right) -$
$- \frac{\partial}{\partial y} \left(\frac{\partial V_z}{\partial x} \right) + \frac{\partial}{\partial z} \left(\frac{\partial V_y}{\partial x} \right) - \frac{\partial}{\partial z} \left(\frac{\partial V_x}{\partial y} \right) = 0$ And as order of taking partial derivatives is not important, it is easy to see that the result of this equation is 0.