

Elasticity

2.1 Deformation modes

If a material is loaded with a force, the atoms within the material are displaced – the material responds with a deformation. This deformation determines the *mechanical behaviour* of the material. Different types of deformation exist which are not only caused by different physical mechanisms, but are also used in different engineering applications. In particular, we distinguish *reversible* deformations, with the deformation disappearing after unloading, and *irreversible* deformations that preserve the deformation after unloading. Reversible deformations are used in springs and vibrating chords; irreversible deformations are employed to produce components, e. g. by forging, or to absorb energy in crash elements. Generally, reversible deformations are called *elastic*, irreversible deformations are called *plastic*.

Different types of deformation can also be distinguished in another way, for they can be either time-dependent or time-independent. A deformation is time-dependent if the material responds with a delay to changes of the load. If – in contrast – the deformation coincides with the change of the load, the deformation is time-independent. Time-dependent deformations are denoted by the prefix *visco-*. Altogether, four different deformation types exist since elastic as well as plastic deformations can be time-dependent or time-independent.

In this chapter, we will start by discussing how external forces and the resulting material deformations can be described. Subsequently, the *time-independent elastic* behaviour of materials will be described. Often, it is simply called ‘*the elastic behaviour*’, although this is not completely correct.

Time-independent plastic deformation will be described in chapters 3, 6, and 8, the time-dependent plastic behaviour is subject of chapters 8 and 11. Time-dependent elastic behaviour is mainly observed in polymers, described in chapter 8.

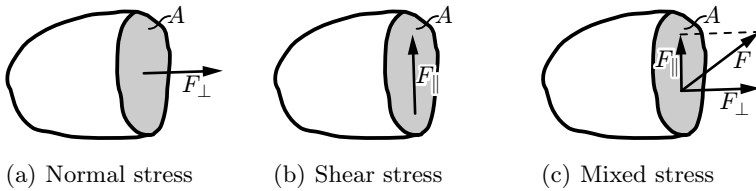


Fig. 2.1. Different stress measures

2.2 Stress and strain

Components used in engineering have strongly varying dimensions and often also a complicated geometry, resulting in loads that vary strongly throughout the component. To dimension components, characteristic parameters for each material are required that describe its mechanical behaviour. These parameters have to be independent of the geometry and dimension of the components so that they can be determined in experiments using standardised specimens. This can be achieved by normalising the load and the deformation on the dimension (area and length, respectively). To describe the varying conditions within a component, the load and deformation measures are specified for small volume elements. Usually, a continuum mechanical approach is used: The investigated scale is large in comparison to the atomic distance. The matter is considered to be distributed continuously, which results in all variables being continuous.

2.2.1 Stress

Components are usually loaded with certain forces or moments. How strong the material is stressed depends on the area loaded. If the area is increased, the stress decreases. The *stress* σ is thus defined as the force divided by the area the force is acting on. Stresses can be distinguished by the relative orientation of the force and the area. If the force F is perpendicular to the area A , the stress

$$\sigma = \frac{F_{\perp}}{A} \quad (2.1)$$

is called a *normal stress* (sometimes also *direct stress*, see figure 2.1(a)). If the force is parallel to the area (figure 2.1(b)), the stress is a *shear stress*

$$\tau = \frac{F_{\parallel}}{A}. \quad (2.2)$$

In all other cases, the force can be decomposed into a normal and a parallel component and normal and shear stresses act simultaneously (figure 2.1(c)).

To describe the loading in a certain point of a material, we imagine it to be cut apart at this point along a *cutting plane*. The stress that was transferred

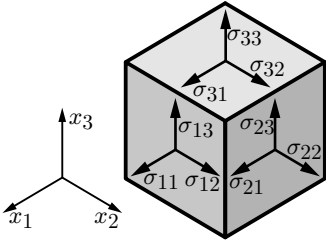


Fig. 2.2. Numbering of the components of the stress tensor $\underline{\underline{\sigma}}$

through this plane by the material cut away now has to be replaced by an external stress vector, the so-called *surface traction*, to retain the equilibrium of force in the material. The value of the surface traction depends on the orientation of the cutting plane. If, for example, we cut a rod loaded with a uniaxial stress σ along a plane perpendicular to the applied force, the surface traction is a vector in the direction of the force with magnitude σ . If the cutting plane is parallel to the force vector, the surface traction vanishes i. e., we don't need to apply a surface traction vector to preserve the equilibrium. The stress state in three dimensions can be determined by cutting along three cutting planes that are preferentially chosen parallel to the coordinate axes. The nomenclature of the stresses is chosen as follows: The first index denotes the normal vector of the cutting plane considered (figure 2.2), the second index denotes the direction of the stress: $\sigma_{ij} = F_j/A_i$.¹ The shear stress on each of the three cutting planes is decomposed into its two components parallel to the coordinate axes. These 9 components of the stress are collected in a component matrix (σ_{ij}) that forms the *stress tensor* of second order $\underline{\underline{\sigma}}$.

In a so-called *classical continuum*, an infinitesimal small material element cannot transfer moments.² From this, it can be shown that

$$\sigma_{ij} = \sigma_{ji} \quad \text{for } i, j = 1 \dots 3 \quad (2.3)$$

holds i. e., the stress tensor is symmetric [67]. It has only 6 independent components, 3 on the diagonal and 3 off-diagonal ones.

If we change the coordinate system, the components of the stress tensor $\underline{\underline{\sigma}}$ (its matrix representation) change, but it still describes the same state of stress. The transformation rules are detailed in appendix A.5.

For any stress tensor $\underline{\underline{\sigma}}$, there is a coordinate system where only the diagonal components of the tensor are non-vanishing, whereas all off-diagonal parts are zero. In this coordinate system, all stresses are thus normal stresses. These stresses are called *principal stresses* of the stress tensor (see appendix A.7); the axes of the coordinate system are called the *principal axes*. Principal stresses are denoted with Roman numerals when they are sorted: $\sigma_I \geq \sigma_{II} \geq \sigma_{III}$;

¹ For shear stresses, τ_{ij} (with $i \neq j$) is frequently used instead.

² This assumption can be relinquished, resulting in the theory of a *Cosserat continuum*. In this case, infinitesimal material elements can transfer moments, resulting in an asymmetric stress tensor.

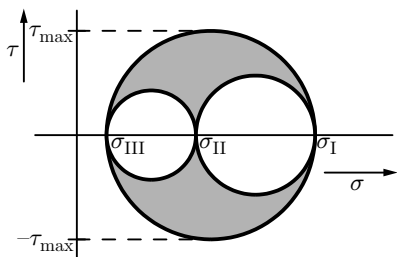


Fig. 2.3. Mohr's circle. Only those stress pairs of the surface traction lying in the grey region can occur.

if they are unsorted, Arabic numerals are used: σ_1 , σ_2 , σ_3 . In its principal coordinate system, the stress tensor is thus simply

$$\underline{\underline{\sigma}} = \begin{pmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{pmatrix}.$$

In many cases (for example when we consider plastic yielding of materials, see section 3.3.2), it is necessary to calculate the shear stresses that can occur in arbitrarily oriented coordinate systems from the known principal stresses. This can be done geometrically with a construction known as *Mohr's circle* [58, 81], see figure 2.3. We draw a diagram with the normal stresses on the abscissa and the shear stresses on the ordinate. The three principal stresses are marked in the diagram and three circles are drawn, each of them bounded by two of the principal stresses. If we cut the material at the point considered, each cutting plane has a certain surface traction which can be decomposed into a pair of a normal (σ) and a shear (τ) component. If we mark all such pairs of σ - τ values for all possible orientations of the cutting plane in the diagram, they form the shaded area in figure 2.3. For instance, there is a cutting plane of maximum shear stress, with a shear stress value of $\tau_{\max} = (\sigma_I - \sigma_{III})/2$ and a normal stress given by the average of the largest and smallest principal stress, $(\sigma_I + \sigma_{III})/2$.

If two principal stresses take the same value, a simple circle without any open area results; if all three principal stresses are identical, the circle degenerates to a point, and the stress state is isotropic.

2.2.2 Strain

If a component is stressed, points within it are displaced. There are different kinds of displacements: The component can be displaced as a whole in a *rigid-body displacement* or it can be rotated rigidly (*rigid-body rotation*). In these cases, distances and angles between points in the material remain unchanged; the component itself is thus still undeformed. To describe the deformation of a component, considering the displacements only is therefore not too helpful. Instead, *changes* of distances and angles between points have to be looked at. This can be done by calculating the change of the displacement with position.

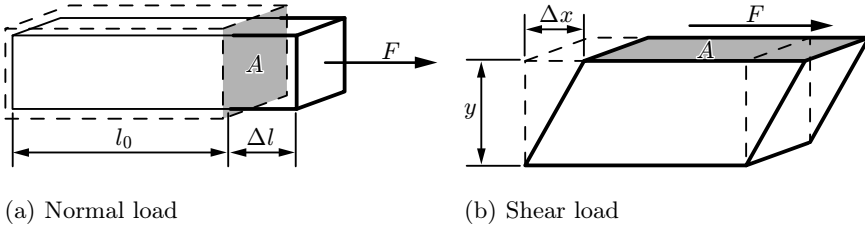


Fig. 2.4. Simple load cases

All deformations, also called *strains*, can be composed from changes in lengths and angles (shearing of the material). To describe changes in length, the *normal* or *direct strain* ε is defined as the difference Δl between the final length l_1 and the initial length l_0 (figure 2.4(a)):

$$\varepsilon = \frac{l_1 - l_0}{l_0} = \frac{\Delta l}{l_0}. \quad (2.4)$$

Changes in the angles are described by the *shear* γ , corresponding to the change in an initially right angle. For small deformations Δx (see figure 2.4(b)), it is defined as

$$\gamma = \frac{\Delta x}{y}, \quad (2.5)$$

with Δx and y being perpendicular.

An arbitrary deformation with small strains³ of a material element can be described – analogous to the stress – by a tensor, the *strain tensor* of second order $\underline{\underline{\varepsilon}}$. To calculate the strain tensor, we chose a coordinate system that is fixed in space and consider the displacement of material points in this system as sketched in figure 2.5. This position-dependent displacement is described by a vector field $\underline{u}(\underline{x})$. To understand how the strain is calculated from the displacement, we first consider some special cases.

A pure strain in normal direction, for example in the x_1 direction, causes the displacement u_1 to increase with increasing x_1 . If we consider two neighbouring points $x_1^{(1)}$ and $x_1^{(2)}$, with an initial, infinitesimal distance $\Delta x_1 \rightarrow 0$, that are displaced by $u_1^{(1)}$ and $u_1^{(2)}$, respectively, the resulting strain is

$$\varepsilon_{11} = \lim_{\Delta x_1 \rightarrow 0} \frac{u_1^{(2)} - u_1^{(1)}}{\Delta x_1} = \frac{\partial u_1}{\partial x_1}.$$

Transferring this result to the other spatial directions, we get for the normal strains

$$\varepsilon_{ii} = \frac{\partial u_i}{\partial x_i}. \quad (2.6)$$

³ Arbitrary deformations with large strains will be discussed in section 3.1.

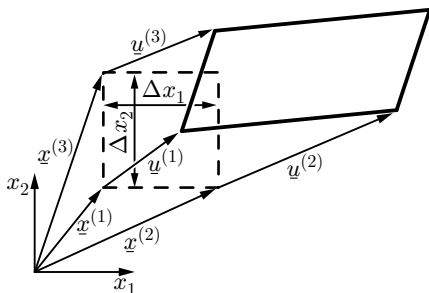


Fig. 2.5. Two-dimensional displacement field in a material. The coordinate system x_i remains fixed in space; the displacements $\underline{u}^{(j)}$ of material elements with the original coordinates $\underline{x}^{(j)}$ refer to the original position

The indices are underscored to denote that the Einstein summation convention is *not* to be used for the repeated index (see appendix A) i. e., they are not summed over.

If the material is sheared, the region considered is distorted and initially right angles are made obtuse or acute. The rotation of the edge parallel to the x_1 axis and of the other edge both contribute to this angular change (cf. figure 2.5). For small rotations and in the limit $\Delta x_1 \rightarrow 0$ and $\Delta x_2 \rightarrow 0$, the resulting shear is

$$\gamma_{12} = \lim_{\Delta x_1 \rightarrow 0} \frac{u_2^{(2)} - u_2^{(1)}}{\Delta x_1} + \lim_{\Delta x_2 \rightarrow 0} \frac{u_1^{(3)} - u_1^{(1)}}{\Delta x_2} = \frac{\partial u_2}{\partial x_1} + \frac{\partial u_1}{\partial x_2}.$$

Generalising to all coordinate axes yields

$$\gamma_{ij} = \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \quad \text{for } i \neq j. \quad (2.7)$$

This definition implies $\gamma_{ji} = \gamma_{ij}$.

Using equations (2.6) and (2.7), all strains can be calculated if they are assumed to be small. However, they cannot be used as components of a tensor, for they do not transform correctly as tensors should. A correct transformation behaviour can be achieved when the shear strain γ_{ij} is replaced by half of its value: $\varepsilon_{ij} = \gamma_{ij}/2$. An additional advantage of this formulation is that equations (2.6) and (2.7) do not have to be written separately for the components, but can be collected in one equation:

$$\varepsilon_{ij} = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right). \quad (2.8)$$

This definition ensures $\varepsilon_{ij} = \varepsilon_{ji}$, rendering the strain tensor symmetric. Similar to the stress tensor, only 6 of its components are independent.

If the material is displaced relative to the coordinate system in a rigid-body translation, the displacement vectors are the same at any material point, $\underline{u}(\underline{x}) = \text{const.}$ This yields $\partial u_i / \partial x_j = 0$ and thus $\varepsilon_{ij} = 0$ as should be expected. This result is intuitively obvious, for a rigid-body translation does not cause strains.

A rigid-body rotation is more problematic. For small rotations around the x_3 axis with an angle α , we find $\partial u_1 / \partial x_1 = \cos \alpha - 1 \approx 0$, $\partial u_2 / \partial x_2 = \cos \alpha - 1 \approx 0$, $\partial u_1 / \partial x_2 = -\sin \alpha \approx -\alpha$ and $\partial u_2 / \partial x_1 = \sin \alpha \approx \alpha$. If we insert this into equation (2.8), the mixed terms $\partial u_1 / \partial x_2$ and $\partial u_2 / \partial x_1$ cancel, resulting in $\varepsilon_{ij} = 0$. However, for large rotations, the approximations are not valid and definition (2.8) is not applicable anymore. Suitable definitions of the strain need more involved tensor calculations and will be discussed in more detail in section 3.1.

2.3 Atomic interactions

In the previous chapter, we saw that different material classes have different types of chemical bonds. The atoms in the materials attract each other by different physical mechanisms. If there were only an attractive force between the atoms, their distance would quickly reduce to zero. However, in addition to the attractive interaction of the atoms, there also is a repulsive one. The repulsive interaction is – in a slightly simplified picture – based on the repulsion of the electron orbitals that cannot penetrate each other. The repulsive interaction is short-ranged i. e., it is only relevant if the distances are small, but for very small distances it becomes much larger than the attractive force.

The distance r between neighbouring atoms (e. g., in a solid) takes a value that minimises the potential energy of the total interaction between the atoms. If we superimpose the repulsive potential $U_R(r)$ and the attractive potential $U_A(r)$, the total potential is

$$U(r) = U_A(r) + U_R(r). \quad (2.9)$$

It is minimised at a stable atomic distance r_0 as sketched in figure 2.6. Usually, atomic distances are between 0.1 nm and 0.5 nm [17]. Due to the shape of the potential, the term *potential well* is frequently used to describe it.

The *interaction force* (or *binding force*) $F_i(r)$ between the atoms can be calculated by differentiating the potential:

$$F_i(r) = -\frac{dU(r)}{dr}. \quad (2.10)$$

In equilibrium, $F_i(r_0) = 0$. If an external force is added to the interaction forces, the stable atomic distance changes and the material deforms.

Because the first derivative of the potential (the negative force) vanishes in equilibrium, the potential energy can be approximated by a spring model

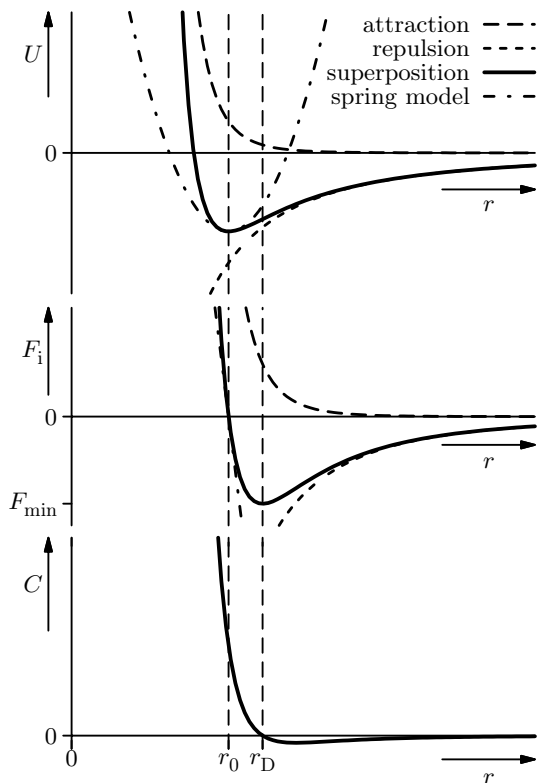


Fig. 2.6. Interaction between two atoms (potential U , binding force $F_i = -dU/dr$, stiffness $C = d^2U/dr^2$)

(a harmonic law) with a spring stiffness k if we are sufficiently close to the equilibrium position r_0 :⁴

$$\begin{aligned}
 U(r) &\approx U_0 + \frac{1}{2}k(r - r_0)^2, \\
 F_i(r) &\approx -k(r - r_0).
 \end{aligned}
 \tag{2.11}$$

The external load on a single bond is equal to the negative internal (binding) force:

$$F \approx k(r - r_0).
 \tag{2.12}$$

Thus, for small displacements, the force is proportional to the displacement.

If the external force is so large that the distance of the atoms attains the value r_D ('D' for 'debonding') shown in figure 2.6 where the restoring force is

⁴ Mathematically, this is a Taylor series cut off at the second-order term.

maximal, a further increase in the external load cannot be borne by the bond. The bond, and thus the material, breaks.

This is also reflected in the stiffness. It decreases from its initial value k at r_0 to zero at r_D and then becomes negative, rendering the bond unstable. If we use the simplifying assumption of a harmonic law to describe the spring, we assume a constant spring stiffness. This is a valid assumption for small displacements, typical for the elastic deformation of metals and ceramics.

2.4 Hooke's law

For small displacements from the equilibrium position, the force between the atoms is proportional to the displacement (see equation (2.12)). This is true not only for a single bond, but also for larger atomic compounds and thus for macroscopic solids. This *linear-elastic behaviour* is described mathematically by *Hooke's law*. It is valid only for small strains. In metals and ceramics, this is not an important constraint because the elastic part of any deformation is small.

For uniaxial loads (figure 2.4(a)), Hooke's law is

$$\sigma = E\varepsilon \quad (2.13)$$

with *Young's modulus* E , also sometimes called the *elastic modulus*. Young's modulus quantifies the stiffness of a material: the larger Young's modulus is, the smaller is the elastic deformation for a given load.

If a component is strained by a strain ε , strains in perpendicular directions also develop. Usually, a positive strain causes a contraction in the transverse direction, justifying the name *transversal contraction* for this phenomenon. It is measured by *Poisson's ratio* ν , defined as

$$\varepsilon_{\text{trans}} = -\nu\varepsilon. \quad (2.14)$$

In many metals, Poisson's ratio is approximately $\nu \approx 0.33$; if the material is incompressible so its volume remains constant, $\nu = 0.5$ holds.

For pure shear (figure 2.4(b)), Hooke's law is

$$\tau = G\gamma,$$

where G is the *shear modulus*. Similar to Young's modulus, the shear modulus quantifies the stiffness of the material in shear.

In elastically isotropic materials, the elastic properties are the same in all spatial directions. In this case, the elastic constants are related as follows:

$$G = \frac{E}{2(1 + \nu)}. \quad (2.15)$$

This equation will be discussed in section 2.4.3.

Table 2.1. Young's modulus of selected materials [8]. For polymers, a more detailed compilation is given in table 8.2

material	E/GPa
metals	$\approx 15 \dots 500$
tungsten	411
nickel alloys	180 ... 234
ferritic steels	200 ... 207
austenitic steels	190 ... 200
cast iron	170 ... 190
copper alloys	120 ... 150
titanium alloys	80 ... 130
brasses and bronzes	103 ... 124
aluminium alloys	69 ... 79
magnesium alloys	41 ... 45
ceramics	$\approx 40 \dots 1000$
diamond	1000
tungsten carbide, WC	450 ... 650
silicon carbide, SiC	450
aluminium oxide, Al_2O_3	390
titanium carbide, TiC	379
magnesium oxide, MgO	250
zirconium monoxide, ZrO	160 ... 241
zirconium dioxide, ZrO_2	145
concrete	45 ... 50
silicon	107
silica glass, SiO_2	94
window glass	69
polymers	$\approx 0.1 \dots 5.0$
polyester	1.0 ... 5.0
nylon	2.0 ... 4.0
polymethylmethacrylate	3.0 ... 3.4
epoxy resins	3.0
polypropylene	0.9
polyethylene	0.2 ... 0.7
composites	
carbon-fibre reinforced polymers	70 ... 200
glass-fibre reinforced polymers	7 ... 45
wood, \parallel to the fibres	9 ... 16
wood, \perp to the fibres	0.6

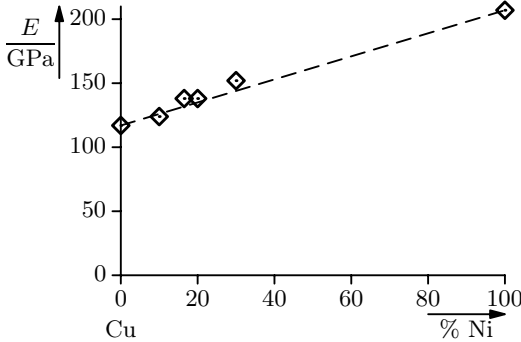


Fig. 2.7. Dependence of Young's modulus on the amount of alloyed nickel in copper [33]

In table 2.1, a survey of the Young's moduli of several engineering materials is given. The elastic stiffness of ceramics slightly exceeds that of metals, but is of the same order of magnitude. Young's modulus of most polymers is much smaller.⁵ This should be expected, for the stiffness is determined by the strength of the atomic bonds, which is larger in ceramics than in metals. In polymers, the weaker inter-molecular bonds determine the stiffness. How Young's modulus can be measured will be described in section 3.2.

From table 2.1, it can also be seen that alloying does not significantly change the stiffness of materials. For example, Young's modulus of different aluminium alloys varies only by about 10%, whereas their strength (see chapter 6) can be raised considerably by alloying.

If two different metals are alloyed, the resulting Young's modulus is not necessarily the weighted average of their two moduli because the binding energy U_{AB} between the atoms A and B is usually not the average of the single-type energies U_{AA} and U_{BB} . Depending on the alloying elements, Young's modulus may even be larger than those of both constituent elements. A rule of thumb is that adding a material with a high melting point (e. g., tungsten to nickel) increases the elastic modulus.

There are a few alloy systems where Young's modulus can be increased considerably. This is the case when both the solubility of the elements and the difference in Young's modulus are large. For example, nickel ($E_{Ni} = 207$ GPa) and copper ($E_{Cu} = 121$ GPa) are completely soluble, and their Young's moduli differ almost by a factor of two. Therefore, Young's modulus of copper-nickel alloys (nickel bronze) can be strongly increased by raising the nickel content (figure 2.7).

Usually, though, these effects are small because the solubility of alloying elements is usually small ($< 10\%$) in technical alloys. Therefore,

⁵ Polymer fibres are an exception, see section 8.5.2.

Young's modulus of most engineering alloys differs only by less than $\pm 10\%$ from that of the un-alloyed matrix. In contrast, the strength, a measure of the maximum load the material can bear, can be strongly increased by alloying and may widely exceed the strength of all alloying elements (see section 6.4).

A particularly efficient way of increasing Young's modulus is to use composites, containing, for example, fibres with large stiffness in a matrix of another material. Composites are the subject of chapter 9.

So far, Hooke's law has only been stated for loads that were either normal or shear loads. In real-world applications, components are usually loaded in a multiaxial state where normal and shear stresses are combined. This case will be considered in section 2.4.2. Afterwards, different cases of special symmetries are considered that allow simplifications of Hooke's law. Prior to this, we will discuss the energy stored in elastic deformations.

2.4.1 Elastic strain energy

Any elastic deformation of a material stores energy as can be easily understood by considering the spring model from section 2.3. To calculate this energy, we consider an (infinitesimal) brick-shaped volume element of length l and cross section A to which a load F is applied. The resulting stress is $\sigma = F/A$. If we increase the stress by an amount $d\sigma$, the external force must increase by $dF = d\sigma A$. The material lengthens by an amount dl .

The work done is $dW = Fdl$.⁶ If we insert $\sigma = F/A$ and the definition of strain, $d\varepsilon = dl/l$, we find for the work done

$$dW = Fdl = \sigma A d\varepsilon l = \sigma d\varepsilon V, \quad (2.16)$$

where $V = Al$ is the volume of the brick. If we normalise the work to the volume, thus switching to the *energy density* $dw = dW/V$, we find $dw = \sigma d\varepsilon$.

The total work done per unit volume in a material strained up to ε_{\max} is the integral over dw :

$$w = \int_0^{\varepsilon_{\max}} \sigma d\varepsilon. \quad (2.17)$$

This equation is valid for arbitrary uniaxial deformations. If the deformation is irreversible, part of the work is transformed to heat and cannot be recovered on unloading. In elastic (reversible) deformations, the energy is stored in

⁶ Here we use the force at the beginning of the strain increment. As we can neglect second-order terms in this infinitesimal calculation, this does not make a difference: $dW = (F + dF)dl = Fdl + dFdl = Fdl$.

the strained atomic bonds and can be recovered.⁷ Because the work done is stored as potential energy of the atomic bonds, the name *elastic potential* is frequently used to describe the stored energy (cf. section 2.3).

This calculation was valid for uniaxial stresses and strains only. For arbitrary stresses and strains, we have to generalise by switching to tensors:

$$w = \int_0^{\underline{\underline{\varepsilon}}_{\max}} \underline{\underline{\sigma}} \cdot \underline{\underline{d\varepsilon}}. \quad (2.18)$$

The product of the stress and the strain increment in this equation is the so-called *double contraction* explained in appendix A.4.

In a linear-elastic material under uniaxial loads, stress and strain are related by Hooke's law, $\sigma = E\varepsilon$. In this case, the integral in equation (2.17) can easily be solved:

$$w^{(\text{el})} = \int_0^{\varepsilon_{\max}} E\varepsilon d\varepsilon = \frac{1}{2}E\varepsilon_{\max}^2 = \frac{1}{2E}\sigma_{\max}^2. \quad (2.19)$$

The elastic strain energy increases quadratically with the stress or the strain (see also exercise 6).

* 2.4.2 Elastic deformation under multiaxial loads⁸

We already saw in section 2.2.2 that a load that causes a normal strain in its direction also causes transversal normal strains. For example, a stress in x_1 direction, σ_{11} , causes the following strains, according to equations (2.13) and (2.14): $\varepsilon_{11} = \sigma_{11}/E$, $\varepsilon_{22} = \varepsilon_{33} = -\nu\sigma_{11}/E$. One component of the stress tensor $\underline{\underline{\sigma}}$ thus acts on several components of the strain tensor $\underline{\underline{\varepsilon}}$. Similarly, a prescribed strain in one direction may change the stresses in other directions. If we restrict ourselves to small deformations, the relation between stress and strain is linear. Mathematically, an arbitrary linear relation between two tensors of second order can be described using a double contraction:

$$\sigma_{ij} = C_{ijkl} \varepsilon_{kl} \quad \text{or} \quad \underline{\underline{\sigma}} = \underset{4}{\mathcal{C}} \cdot \underline{\underline{\varepsilon}} \quad (2.20)$$

The *elasticity tensor* $\underset{4}{\mathcal{C}}$ is a tensor of fourth order. It can be considered as a four-dimensional 'matrix' with three components in each of its 4 directions. Its $3^4 = 81$ components C_{ijkl} are the material parameters that completely describe the (linear) elastic behaviour.

Because the stress and the strain tensor contain only 6 independent components each, due to their symmetry, the elasticity tensor $\underset{4}{\mathcal{C}}$ needs only $6^2 = 36$ independent parameters.

⁷ The storage and dissipation of energy is also discussed in exercise 26.

⁸ Sections with a title marked by a * contain advanced information which can be skipped without impairing the understanding of subsequent topics.

That not all 81 components of the elasticity tensor are needed can be most easily understood using an example. For σ_{12} , we find from equation (2.20)

$$\begin{aligned}\sigma_{12} &= C_{1211} \varepsilon_{11} + C_{1212} \varepsilon_{12} + C_{1213} \varepsilon_{13} \\ &\quad + C_{1221} \varepsilon_{21} + C_{1222} \varepsilon_{22} + C_{1223} \varepsilon_{23} \\ &\quad + C_{1231} \varepsilon_{31} + C_{1232} \varepsilon_{32} + C_{1233} \varepsilon_{33}.\end{aligned}$$

Using the symmetry condition $\varepsilon_{ij} = \varepsilon_{ji}$, we can collect terms as follows:

$$\begin{aligned}\sigma_{12} &= C_{1211} \varepsilon_{11} + C_{1222} \varepsilon_{22} + C_{1233} \varepsilon_{33} \\ &\quad + (C_{1212} + C_{1221}) \varepsilon_{12} \\ &\quad + (C_{1213} + C_{1231}) \varepsilon_{13} \\ &\quad + (C_{1223} + C_{1232}) \varepsilon_{23}.\end{aligned}$$

The components C_{ijkl} and C_{ijlk} always appear together and thus represent only one independent parameter. This can be implemented by using the condition $C_{ijkl} = C_{ijlk}$. Thus, the 9 components C_{12kl} reduce to only 6 independent components C_{1211} , C_{1222} , C_{1233} , C_{1212} , C_{1213} , and C_{1223} .

Furthermore, because $\sigma_{12} = \sigma_{21}$, we can also set $C_{ijkl} = C_{jikl}$. The two symmetry conditions $C_{ijkl} = C_{jikl}$ and $C_{ijkl} = C_{ijlk}$ reduce the number of independent components of the elasticity tensor to 36.

The reduced number of components enables us to use a simplified matrix notation (*Voigt notation*), rewriting the tensors of second order as column matrices and the tensor of fourth order as a quadratic matrix: $(\sigma_{ij}) \rightarrow (\sigma_\alpha)$, $(\varepsilon_{ij}) \rightarrow (\varepsilon_\alpha)$, and $(C_{ijkl}) \rightarrow (C_{\alpha\beta})$. The new Greek indices α and β take values from 1 to 6. Writing down the components explicitly, we have

$$\begin{aligned}(\sigma_\alpha) &= \left(\sigma_{11} \quad \sigma_{22} \quad \sigma_{33} \quad \sigma_{23} \quad \sigma_{13} \quad \sigma_{12} \right)^T, \\ (\varepsilon_\alpha) &= \left(\varepsilon_{11} \quad \varepsilon_{22} \quad \varepsilon_{33} \quad \gamma_{23} \quad \gamma_{13} \quad \gamma_{12} \right)^T\end{aligned}$$

with $\gamma_{ij} = 2\varepsilon_{ij}$. The factors of 2 for the mixed components are due to the re-writing of the tensor components.

This can again be understood most easily using an example. The stress component σ_{11} is, according to equation (2.20),

$$\begin{aligned}\sigma_{11} &= C_{1111} \varepsilon_{11} + C_{1112} \varepsilon_{12} + C_{1113} \varepsilon_{13} \\ &\quad + C_{1121} \varepsilon_{21} + C_{1122} \varepsilon_{22} + C_{1123} \varepsilon_{23} \\ &\quad + C_{1131} \varepsilon_{31} + C_{1132} \varepsilon_{32} + C_{1133} \varepsilon_{33}.\end{aligned}$$

With help of the symmetry conditions $\varepsilon_{21} = \varepsilon_{12}$, $\varepsilon_{31} = \varepsilon_{13}$, $\varepsilon_{32} = \varepsilon_{23}$, $C_{1121} = C_{1112}$, $C_{1131} = C_{1113}$, and $C_{1132} = C_{1123}$, we find

$$\begin{aligned}\sigma_{11} &= C_{1111}\varepsilon_{11} + C_{1122}\varepsilon_{22} + C_{1133}\varepsilon_{33} \\ &\quad + 2C_{1123}\varepsilon_{23} + 2C_{1113}\varepsilon_{13} + 2C_{1112}\varepsilon_{12}.\end{aligned}$$

The sequence of the mixed terms is not universally agreed upon, but a consistent convention has to be used in any calculation.⁹

The elasticity tensor possesses further symmetries due to the existence of an elastic potential [108]. The elasticity matrix $(C_{\alpha\beta})$ is symmetric because of this and the number of independent components reduces further to 21 (6 diagonal and 15 off-diagonal ones).

The elastic potential was already introduced in equation (2.18). Writing it in differential form yields $dw = \underline{\underline{\sigma}} \cdot \underline{\underline{d\varepsilon}}$, or, after re-writing,

$$\sigma_{ij} = \frac{dw}{d\varepsilon_{ij}} \quad \text{or} \quad \underline{\underline{\sigma}} = \frac{dw}{d\underline{\underline{\varepsilon}}}.$$

Thus, the stress tensor can be calculated by differentiating the elastic potential with respect to the strains.

Hooke's law, equation (2.20), can also be written in differential form:

$$C_{ijkl} = \frac{\partial\sigma_{ij}}{\partial\varepsilon_{kl}} \quad \text{or} \quad \underset{4}{C} = \frac{\partial\underline{\underline{\sigma}}}{\partial\underline{\underline{\varepsilon}}}.$$

The elasticity tensor is thus the derivative of the stress with respect to the strain.

Inserting the stress from the previous equation, we find

$$C_{ijkl} = \frac{\partial^2 w}{\partial\varepsilon_{ij}\partial\varepsilon_{kl}} \quad \text{or} \quad \underset{4}{C} = \frac{\partial^2 w}{\partial\underline{\underline{\varepsilon}}\partial\underline{\underline{\varepsilon}}}.$$

Because the sequence of taking the derivatives is arbitrary, we find the symmetry condition $C_{ijkl} = C_{klij}$ for the elasticity tensor, or, for the elasticity matrix, $(C_{\alpha\beta}) = (C_{\beta\alpha})$.

Altogether, the three symmetry conditions $C_{ijkl} = C_{jikl} = C_{ijlk} = C_{klij}$ reduce the number of independent components to 21 even in anisotropic materials.

Writing out all components, Hooke's law looks like this:

⁹ When working with material parameters, the convention in use has to be checked carefully.

$$\begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{pmatrix} = \begin{pmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{12} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{13} & C_{23} & C_{33} & C_{34} & C_{35} & C_{36} \\ C_{14} & C_{24} & C_{34} & C_{44} & C_{45} & C_{46} \\ C_{15} & C_{25} & C_{35} & C_{45} & C_{55} & C_{56} \\ C_{16} & C_{26} & C_{36} & C_{46} & C_{56} & C_{66} \end{pmatrix} \begin{pmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \gamma_{23} \\ \gamma_{13} \\ \gamma_{12} \end{pmatrix}. \quad (2.21)$$

This notation is easier to handle than the tensor notation. Its disadvantage is that coordinate transformations cannot be performed; in this case, the tensor notation must be used.

The arrangement of atoms in a crystal lattice causes further symmetry conditions that will be discussed in the next sections.

* 2.4.3 Isotropic material

A material is mechanically *isotropic* if all of its mechanical properties are the same in all spatial directions. The elasticity tensor must thus remain unchanged by arbitrary rotations of the material or the coordinate system. Its components must be invariant with respect to rotations.

This invariance property can be used to show that the elasticity matrix has the simple form:

$$(C_{\alpha\beta}) = \begin{pmatrix} C_{11} & C_{12} & C_{12} & & & \\ C_{12} & C_{11} & C_{12} & & & \\ C_{12} & C_{12} & C_{11} & & & \\ & & & C_{44} & & \\ & & & & C_{44} & \\ & & & & & C_{44} \end{pmatrix} \quad (2.22)$$

with the additional relation

$$C_{44} = \frac{C_{11} - C_{12}}{2}. \quad (2.23)$$

All components not specified vanish, so there are only two independent parameters, C_{11} and C_{12} .

The following relations between these parameters and the more familiar Young's modulus E , Poisson's ratio ν , and shear modulus G hold:

$$\begin{aligned} C_{11} &= \frac{E(1-\nu)}{(1+\nu)(1-2\nu)}, \\ C_{12} &= \frac{E\nu}{(1+\nu)(1-2\nu)}, \\ C_{44} = G &= \frac{E}{2(1+\nu)}. \end{aligned} \quad (2.24)$$

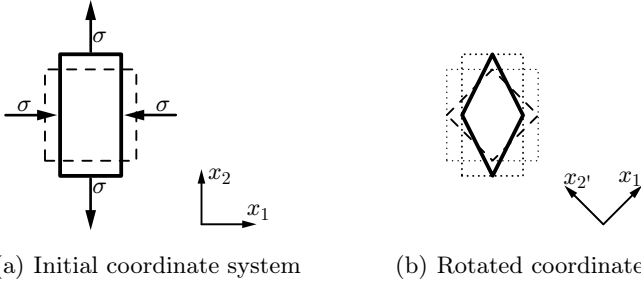


Fig. 2.8. Example to demonstrate the isotropy equation (2.23): Illustration of the loading and the resulting deformation. Both figures show the same deformation, merely viewed in different coordinate systems

Thus, the σ_{11} component is

$$\sigma_{11} = \frac{E}{(1 + \nu)(1 - 2\nu)} \left((1 - \nu)\varepsilon_{11} + \nu(\varepsilon_{22} + \varepsilon_{33}) \right) \quad (2.25a)$$

and σ_{12} is given by

$$\sigma_{12} = G\gamma_{12}. \quad (2.25b)$$

Apart from E , G , and ν , the so-called *Lamé's elastic constants* λ and μ are sometimes used. Their relation to the other elastic constants is as follows [16, 112]:

$$\lambda = C_{12} = \frac{E\nu}{(1 + \nu)(1 - 2\nu)},$$

$$\mu = C_{44} = \frac{E}{2(1 + \nu)}.$$

From equation (2.23), we find $C_{11} = \lambda + 2\mu$.

The validity of the condition (2.23) can be illustrated using the following example.¹⁰ A material is deformed in plane strain with the following strain tensor

$$(\varepsilon_{ij}) = \begin{pmatrix} -\varepsilon & 0 & 0 \\ 0 & \varepsilon & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

written in the x_i coordinate system (see figure 2.8(a)). Using Hooke's law (2.21) and the elasticity matrix from equation (2.22), we find for the required stress

¹⁰ The calculation is further elaborated in exercise 5.

$$(\sigma_{ij}) = \begin{pmatrix} -\varepsilon(C_{11} - C_{12}) & 0 & 0 \\ 0 & \varepsilon(C_{11} - C_{12}) & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (2.26)$$

If we consider the same deformation in a coordinate system $x_{i'}$ that is rotated by 45° relative to the x_i system, the coordinate transformation results in the following strain tensor:

$$(\varepsilon_{i'j'}) = \begin{pmatrix} 0 & \varepsilon & 0 \\ \varepsilon & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (2.27)$$

This corresponds to pure shear with $\gamma_{12} = 2\varepsilon$, see figure 2.8(b). If we ignore the isotropy of the elasticity tensor for a moment, we have to assume that its components are different in different coordinate systems. In the primed coordinate system, $\sigma_{\alpha'} = C_{\alpha'\beta'} \varepsilon_{\beta'}$ leads to

$$(\sigma_{i'j'}) = \begin{pmatrix} 0 & 2\varepsilon C_{A'A'} & 0 \\ 2\varepsilon C_{A'A'} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (2.28)$$

The stresses (σ_{ij}) and $(\sigma_{i'j'})$ describe the same state of stress. Therefore, a coordinate transformation must transform (σ_{ij}) to $(\sigma_{i'j'})$:

$$(\sigma_{i'j'}) = \begin{pmatrix} 0 & 2\varepsilon(C_{11} - C_{12}) & 0 \\ 2\varepsilon(C_{11} - C_{12}) & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (2.29)$$

Comparing the components in equation (2.28) and (2.29), we find

$$C_{A'A'} = \frac{C_{11} - C_{12}}{2}. \quad (2.30)$$

Because the material is isotropic, $C_{\alpha'\beta'} = C_{\alpha\beta}$ and, especially, $C_{A'A'} = C_{44}$. Thus, equation (2.30) is the same as (2.23).

Frequently, Hooke's law is not needed to calculate the stress components from a given strain, as in equation (2.20), but to determine the strains from the stresses. We can rearrange equation (2.20) as follows:

$$\varepsilon_{ij} = S_{ijkl} \sigma_{kl}. \quad (2.31)$$

S_{ijkl} is the *compliance tensor*, the inverse of the elasticity tensor C_{ijkl} .¹¹ Because inverting a matrix is an awkward calculation, the components of the compliance matrix are written explicitly here:

¹¹ We can also invert the elasticity matrix in the Voigt notation instead: $(S_{\alpha\beta}) = (C_{\alpha\beta})^{-1}$.

$$(S_{\alpha\beta}) = \begin{pmatrix} 1/E & -\nu/E & -\nu/E & & & \\ -\nu/E & 1/E & -\nu/E & & & \\ -\nu/E & -\nu/E & 1/E & & & \\ & & & 1/G & & \\ & & & & 1/G & \\ & & & & & 1/G \end{pmatrix}. \quad (2.32)$$

Again, there is an additional condition, $S_{44} = 2(S_{11} - S_{12})$, from which we can derive equation (2.15), $G = E/2(1 + \nu)$. Inserting equation (2.32) into Hooke's law, we find for the ε_{11} component, for instance,

$$\varepsilon_{11} = \frac{1}{E}(\sigma_{11} - \nu(\sigma_{22} + \sigma_{33})) \quad (2.33a)$$

and for the γ_{12} component

$$\gamma_{12} = \frac{1}{G}\sigma_{12}. \quad (2.33b)$$

The other components are analogous.

If we take a closer look at the elasticity matrix ($C_{\alpha\beta}$), equation (2.22), and the compliance matrix ($S_{\alpha\beta}$), equation (2.32), we realise the following pattern: Both are of the form

$$\begin{pmatrix} \bullet & \bullet & \bullet & | & & \\ \bullet & \bullet & \bullet & | & & \\ \bullet & \bullet & \bullet & | & & \\ \hline & & & | & \bullet & \\ & & & | & \bullet & \\ & & & | & \bullet & \bullet \end{pmatrix},$$

where a \bullet marks a number and unoccupied spaces mark zero values. The upper right and lower left sub-matrices describe the relation between shear stresses and normal strains and between normal stresses and shear strains. As they are vanishing, there is no coupling between those components. Therefore, in a fixed coordinate system, normal stresses cannot cause shear strains and shear stresses cannot cause normal strains in an isotropic material.

The lower right sub-matrix, relating shear stresses and shear strains, is diagonal. Shear stresses thus can only cause shear strains of the same orientation.

The upper left sub-matrix, which relates normal stresses and normal strains, is fully occupied. Therefore, a normal stress induces not only a strain in the same direction, but also transverse normal strains, the transverse contraction. Similarly, a normal strain causes stresses in transverse directions.

The consequences of these couplings between the different components can be illustrated using an example. We want to calculate the stiffness in x_1 direction of a component for two different cases. In the first case, the component can deform freely in the x_2 and x_3 direction, so the resulting

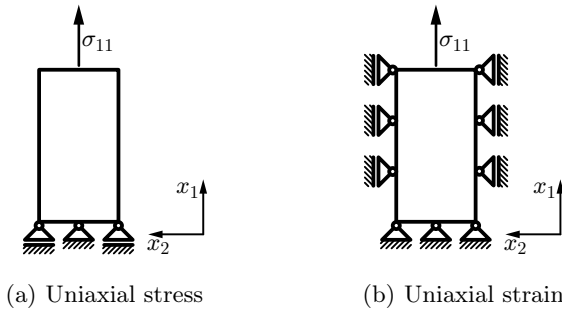


Fig. 2.9. Two different constraints on the same component

stress state is uniaxial and $\sigma_{22} = \sigma_{33} = 0$ (figure 2.9(a)). In the second case, transversal contractions are suppressed, $\varepsilon_{22} = \varepsilon_{33} = 0$, and the state is one of uniaxial strain (figure 2.9(b)).

For uniaxial stresses, it is easiest to calculate the strains with equation (2.31). This yields

$$(\varepsilon_{ij}) = \begin{pmatrix} \sigma_{11}/E & 0 & 0 \\ 0 & -\sigma_{11}\nu/E & 0 \\ 0 & 0 & -\sigma_{11}\nu/E \end{pmatrix}.$$

In x_1 direction, we thus find the uniaxial Hooke's law (2.13), $\sigma_{11} = E\varepsilon_{11}$.

In the case of uniaxial strain, equation (2.20) can be employed, resulting in

$$(\sigma_{ij}) = \begin{pmatrix} C_{11}\varepsilon_{11} & 0 & 0 \\ 0 & C_{12}\varepsilon_{11} & 0 \\ 0 & 0 & C_{12}\varepsilon_{11} \end{pmatrix}.$$

In x_1 direction, we find by using equation (2.24)

$$\sigma_{11} = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)}\varepsilon_{11}.$$

If we assume a Poisson's ratio of $\nu = 1/3$, we get

$$\sigma_{11} = \frac{3}{2}E\varepsilon_{11}.$$

By suppressing transverse contractions, the stiffness of the component increases by 50% compared to the uniaxial stress state. This example also illustrates that the simple relation $\sigma = E\varepsilon$ must not be used inconsiderately, even if only the stresses and strains in one direction are of interest.

* 2.4.4 Cubic lattice

In a cubic crystal, the material properties are anisotropic, but there are a number of rotational symmetries. For example, rotations by multiples of 90°

around the $\langle 100 \rangle$ axes¹² do not change the crystal relative to the coordinate system. Further symmetries are rotations by multiples of 120° around the $\langle 111 \rangle$ axes and by multiples of 180° around the $\langle 110 \rangle$ axes. All these rotations must leave the elasticity tensor and the compliance tensor invariant. Using tensor algebra, the elasticity matrix can be shown to have the following form in a coordinate system parallel to the edges of the unit cell:

$$(S_{\alpha\beta}) = \begin{pmatrix} S_{11} & S_{12} & S_{12} & & & \\ S_{12} & S_{11} & S_{12} & & & \\ S_{12} & S_{12} & S_{11} & & & \\ & & & S_{44} & & \\ & & & & S_{44} & \\ & & & & & S_{44} \end{pmatrix}. \quad (2.34)$$

Unspecified components vanish. Thus, the three independent constants S_{11} , S_{12} , and S_{44} remain. If the coordinate system is not parallel to the edges of the unit cell, a coordinate transformation of the elasticity tensor has to be used to find the components. In this case, the elasticity matrix takes a shape different from that in equation (2.34).

Because the material properties are direction-dependent in a cubic crystal, they have to be stated together with the corresponding direction. According to the definition, the load direction has to be stated for Young's modulus: E_i . Because the shear stress τ_{ij} and shear strain γ_{ij} have two indices, two indices are needed for the shear modulus G_{ij} . Poisson's ratio relates strains in two directions. Here the second index 'j' denotes the direction of the strain that causes the transversal contraction in the direction marked by the first index 'i': $\varepsilon_{ii} = -\nu_{ij}\varepsilon_{jj}$.¹³ If the coordinate system is aligned with the axes of the unit cell, the directions can be characterised using Miller indices, for example $E_{\langle 100 \rangle}$. The following relations between the components S_{ij} and E , G , and ν hold:

$$\begin{aligned} S_{11} &= \frac{1}{E_{\langle 100 \rangle}}, \\ S_{12} &= -\frac{\nu_{\langle 010 \rangle \langle 100 \rangle}}{E_{\langle 100 \rangle}} = -\frac{\nu_{\langle 001 \rangle \langle 100 \rangle}}{E_{\langle 100 \rangle}}, \\ S_{44} &= \frac{1}{G_{\langle 010 \rangle \langle 100 \rangle}} = \frac{1}{G_{\langle 001 \rangle \langle 100 \rangle}}. \end{aligned} \quad (2.35)$$

$\langle 100 \rangle$ is the set of all directions that are parallel to the edges of the unit cell. In cubic crystals, it is rather unusual to work with E , G , and ν . Instead, the components S_{11} , S_{12} , and S_{44} of the compliance matrix or C_{11} , C_{12} , and C_{44} of the elasticity matrix are used.

¹² Directions and planes in crystals are described using *Miller indices*, explained in appendix B.

¹³ As before, underscoring the indices indicates that no summation over this repeated index is done, see appendix A.4.

There is no equation similar to (2.23) in a cubic crystal; S_{11} , S_{12} , and S_{44} (or C_{11} , C_{12} , and C_{44}) are not related. This can be seen from the example from section 2.4.3 on page 47. Up to equation (2.30), $C_{4'4'} = (C_{11} - C_{12})/2$, the calculation remains unchanged. If the material is anisotropic, as in the case of a cubic crystal, $C_{4'4'} \neq C_{44}$, so

$$C_{44} \neq \frac{C_{11} - C_{12}}{2}.$$

It is sufficient to know the elastic constants in one coordinate system (for example, S_{11} , S_{12} , and S_{44}) to calculate the properties in any other coordinate system.

To do this, we have to transform $\underline{\underline{C}}$ or $\underline{\underline{S}}$ to the desired coordinate system. The transformation has to be done using the tensors, not the matrices $\underline{\underline{C}}$ or $\underline{\underline{S}}$ in the simplified Voigt notation, because these matrices do not transform correctly.

Young's modulus in arbitrary directions $[hkl]$, for instance, follows the relation

$$\frac{1}{E_{[hkl]}} = S_{11} - [2(S_{11} - S_{12}) - S_{44}](\alpha^2\beta^2 + \alpha^2\gamma^2 + \beta^2\gamma^2) \quad (2.36)$$

with $\alpha = \cos([hkl], [100])$, $\beta = \cos([hkl], [010])$, and $\gamma = \cos([hkl], [001])$.

The *anisotropy factor* A quantifies the difference of the mechanical behaviour relative to an isotropic material. It is defined as

$$A = \frac{2(S_{11} - S_{12})}{S_{44}}. \quad (2.37)$$

If $A = 1$, the material is isotropic, otherwise it is anisotropic.

In the elasticity matrix $(C_{\alpha\beta})$, the same components are occupied as in the compliance matrix $(S_{\alpha\beta})$.¹⁴ Both matrices can be converted using the following equations which are also valid for an isotropic material:

$$C_{11} = \frac{S_{11} + S_{12}}{(S_{11} - S_{12})(S_{11} + 2S_{12})}, \quad (2.38a)$$

$$C_{12} = -\frac{S_{12}}{(S_{11} - S_{12})(S_{11} + 2S_{12})}, \quad (2.38b)$$

$$C_{44} = \frac{1}{S_{44}} \quad (2.38c)$$

and

$$S_{11} = \frac{C_{11} + C_{12}}{(C_{11} - C_{12})(C_{11} + 2C_{12})}, \quad (2.39a)$$

¹⁴ As long as the coordinate system is parallel to the edges of the unit cell.

$$S_{12} = -\frac{C_{12}}{(C_{11} - C_{12})(C_{11} + 2C_{12})}, \quad (2.39b)$$

$$S_{44} = \frac{1}{C_{44}}. \quad (2.39c)$$

The considerations concerning the coupling between different stress and strain components from the end of section 2.4.3 apply also to cubic crystals.

* 2.4.5 Orthorhombic crystals and orthotropic elasticity

The unit cell of the orthorhombic crystal is brick-shaped. The elastic properties are therefore symmetric with respect to three perpendicular planes. In a coordinate system that is parallel to the edges of the unit cell, the compliance matrix (equation (2.31)) takes the form

$$(S_{\alpha\beta}) = \begin{pmatrix} S_{11} & S_{12} & S_{13} & & & \\ S_{12} & S_{22} & S_{23} & & & \\ S_{13} & S_{23} & S_{33} & & & \\ & & & S_{44} & & \\ & & & & S_{55} & \\ & & & & & S_{66} \end{pmatrix} \\ = \begin{pmatrix} 1/E_1 & -\nu_{12}/E_2 & -\nu_{13}/E_3 & & & \\ -\nu_{21}/E_1 & 1/E_2 & -\nu_{23}/E_3 & & & \\ -\nu_{31}/E_1 & -\nu_{32}/E_2 & 1/E_3 & & & \\ & & & 1/G_{23} & & \\ & & & & 1/G_{13} & \\ & & & & & 1/G_{12} \end{pmatrix}. \quad (2.40)$$

Again, the unspecified components vanish. Altogether, there are nine independent elastic constants. It has to be noted that the compliance tensor is symmetric, so some parameters are related, for example $-\nu_{21}/E_1 = -\nu_{12}/E_2$. Nevertheless, it is useful to discriminate between ν_{12} and ν_{21} , for they are defined by transversal contraction.

In a coordinate system parallel to the edges of the unit cell, normal stresses can only cause normal strains, and shear stresses only shear strains. This is not valid anymore if the coordinate system is arbitrarily oriented, so normal strain and shear are coupled.

The orthorhombic crystal lattice itself is not too important technically because there are only a small number of materials crystallising in this structure. Composites (chapter 9), however, frequently have the same symmetry because they may contain aligned fibres. Materials with the same symmetry as an orthorhombic crystal are called *orthotropic*.

Table 2.3. Elastic constants of different single crystals [35,98,105]. $E_{\text{isotr.}}$ is Young's modulus of a nearly isotropic polycrystal

cubic materials							
material	$E_{\text{isotr.}}$ GPa	$E_{\langle 100 \rangle}$ GPa	$E_{\langle 111 \rangle}$ GPa	A	C_{11} GPa	C_{12} GPa	C_{44} GPa
metals and semi-metals							
Al	70	64	76	1.23	108	61	29
Au	78	43	117	1.89	186	157	42
Cu	121	67	192	3.22	168	121	75
α -Fe	209	129	276	2.13	233	124	117
Ni	207	137	305	2.50	247	147	125
Si	–	130	188	1.57	166	64	80
W	411	411	411	1.00	501	198	151
ceramics							
diamond	–	1050	1200	1.20	1076	125	576
MgO	310	247	343	1.54	291	90	155
NaCl	37	44	32	0.72	49	13	13
TiC	–	476	429	0.88	512	110	117
hexagonal materials							
material	$E_{\text{isotr.}}$ GPa	C_{11} GPa	C_{33} GPa	C_{44} GPa	C_{12} GPa	C_{13} GPa	
Mg	44	60	62	16	26	22	
Ti	112	162	181	47	92	69	
Zn	103	164	64	39	36	53	

* 2.4.7 Other crystal lattices

The number of independent elastic parameters can also be determined for the other crystal lattices and is listed in table 2.2. Generally, couplings between shear stresses and normal strains and normal stresses and shear strains can occur when the number of independent parameters is larger than three. In this case, a uniaxial stress can cause not only normal strains, but also shear strains as we already saw for the example of the orthorhombic crystal.

* 2.4.8 Examples

Table 2.3 contains an overview of the elastic constants for some metals and ceramics. As can be seen, the anisotropy factor of tungsten is 1.0, so it is (almost) isotropic even as a single crystal. For most other materials, almost isotropic properties can only be found in a polycrystalline state. The direction dependence of Young's modulus for selected materials is plotted in figure 2.10.

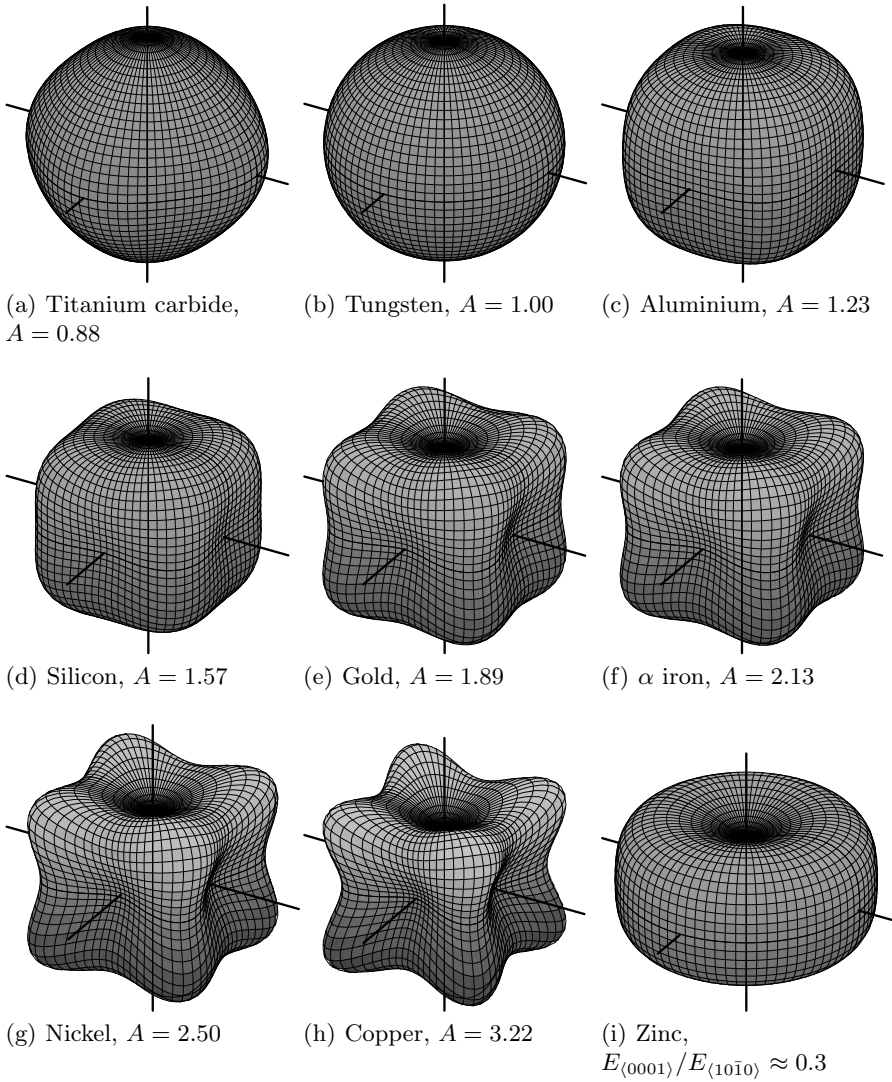


Fig. 2.10. Orientation dependence of Young's modulus for some materials of table 2.3. In each spatial direction, the distance of the surface from the origin is a measure of Young's modulus

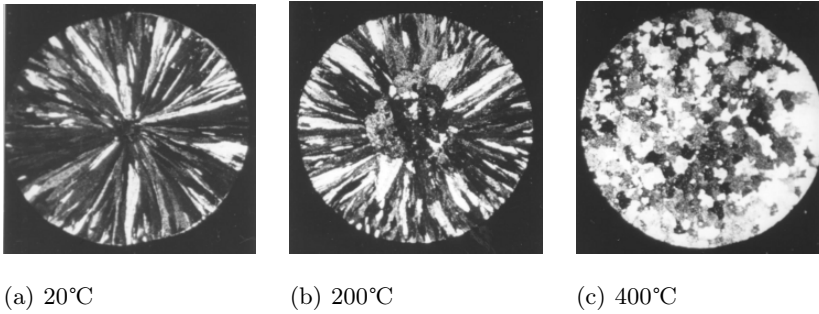


Fig. 2.11. Microstructure of technical pure aluminium for different mould temperatures (cast temperature 900°C). The resulting preferential crystal orientation is the more pronounced, the colder the mould is

* 2.5 Isotropy and anisotropy of macroscopic components

Single crystals are usually mechanically anisotropic as we saw in the preceding sections. In a polycrystalline material, the grains are frequently oriented randomly, and the mechanically anisotropic effects are evened out macroscopically. The material is thus approximately isotropic.

However, there are some cases where a macroscopic component can be anisotropic:

- The component consists of a single crystal. One example are turbine blades used at extreme thermal loads (see also page 58).
- The grains are not small compared to the dimensions of the component itself, so there is insufficient averaging.
- The material is a composite with preferred orientation of the reinforcing phase. Fibre composites are the most important example (see chapter 9).
- During solidification or recrystallisation, a *texture* is formed in the material i. e., the grains have a preferential orientation. This may be due to thermal gradients during solidification of an alloy (see figure 2.11): Solidification starts at the coldest point with the formation of a large number of small nuclei that grow in the direction of the temperature gradient. The speed of crystal growth depends on the crystal orientation, resulting in some grains overtaking the others. The final crystal structure is transversally isotropic. This process can be exploited technically to manufacture directionally solidified materials. One example are turbine blades containing very long grains oriented in the longitudinal direction of the blade (see figure 2.12). Why this is done is explained in the next section.
- The grains have rotated due to large plastic deformations ($> 50\%$), producing a textured material. The reason for this orientation is that crystals

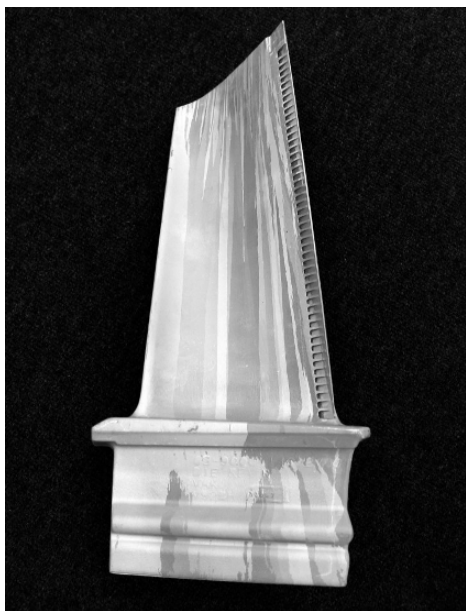


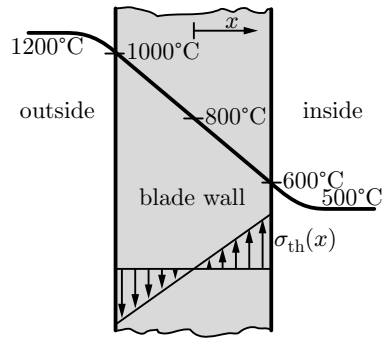
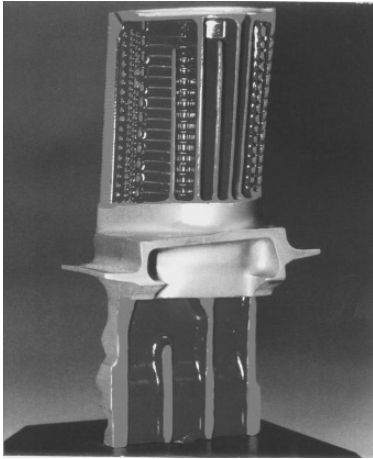
Fig. 2.12. Directionally solidified gas turbine blade. The grains are made visible by etching. Some grains extend over the total length of the blade (385 mm). Courtesy of Siemens AG, Power Generation, Mülheim, Germany

can only deform plastically in certain planes and directions.¹⁶ Deformations of this magnitude are frequently encountered in metal working, for example drawing or rolling.

* How to exploit the elastic anisotropy: Gas turbine blades

Gas turbine blades (figure 2.13(a)) are facing extreme conditions: They have to withstand large mechanical loads due to centrifugal forces at high temperature. To at least partly protect the material from the extreme gas temperatures of 1200°C or more, the blades are cooled from the inside with air of about 500°C. If the wall of the turbine blade has a thickness of about 2 mm and is exposed to the process gas with a temperature of 1200°C, its surface temperature will be about $T_{\text{out}} = 1000^\circ\text{C}$, whereas on the inside it is only $T_{\text{in}} = 600^\circ\text{C}$ (figure 2.13(b)). Due to thermal expansion, the material would expand on the outside, but is partly constrained by the cooler inside wall. Thus, large compressive thermal stresses form on the outside and tensile stresses on the inside. In the middle of the wall, there will be a neutral axis at about $T_{\text{m}} =$

¹⁶ We will discuss this in chapter 6.



(a) Cut-away view. Courtesy of Lufthansa Technik AG, Hamburg, Germany (b) Temperature distribution in the wall

Fig. 2.13. Gas turbine blade of a jet engine. Cooling channels inside the blade are used to air-cool the blade

800°C where thermal stresses vanish. The thermal stress σ_{th} at any point x can be calculated approximately by

$$\sigma_{th}(x) = E \varepsilon_{th} = E \alpha (T_m - T(x)). \tag{2.42}$$

Here $T(x)$ is the local temperature. The thermal stress is thus proportional to the *coefficient of thermal expansion* α and to Young’s modulus E . If we can reduce Young’s modulus in the direction of the thermal stresses, the stresses are reduced, thus either increasing the stress tolerance or allowing to raise the temperature and thus the efficiency of the turbine. In this context, it is irrelevant that the elastic deformations due to centrifugal loads increase when E is reduced, for they are small enough not to compromise the component in any case.

If we assume, as an example, a turbine blade made of a polycrystalline, isotropic nickel-base superalloy with Young’s modulus $E_{isotr.} = 200\,000$ MPa and a coefficient of thermal expansion of $\alpha = 15 \times 10^{-6} \text{ K}^{-1}$, we can estimate the stresses at the outside to $\sigma_{th,out} = -600$ MPa and those at the inside to $\sigma_{th,in} = 600$ MPa.

Now we manufacture the turbine blade from a single crystal or a directionally solidified material oriented in the $\langle 100 \rangle$ direction with Young’s modulus of $E_{\langle 100 \rangle} = 135\,000$ MPa. The thermal stresses at the same temperature are now $\sigma_{th,out,\langle 100 \rangle} = -405$ MPa, $\sigma_{th,in,\langle 100 \rangle} = 405$ MPa. If we assume that the maximum stress the material can bear is 600 MPa, we can raise the surface temperature to almost 1100°C without having to change the material.

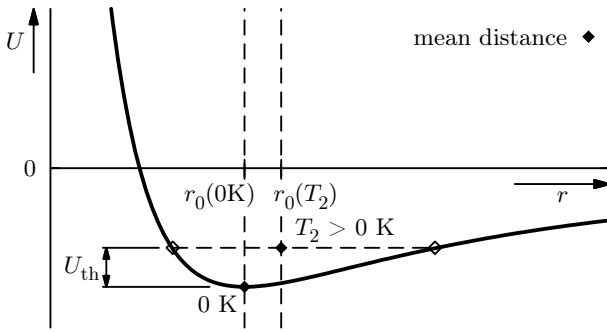


Fig. 2.14. Interaction potential between two atoms. When the temperature is increased, additional thermal energy U_{th} is available. The asymmetry of the potential well causes an increase of the average atomic distance

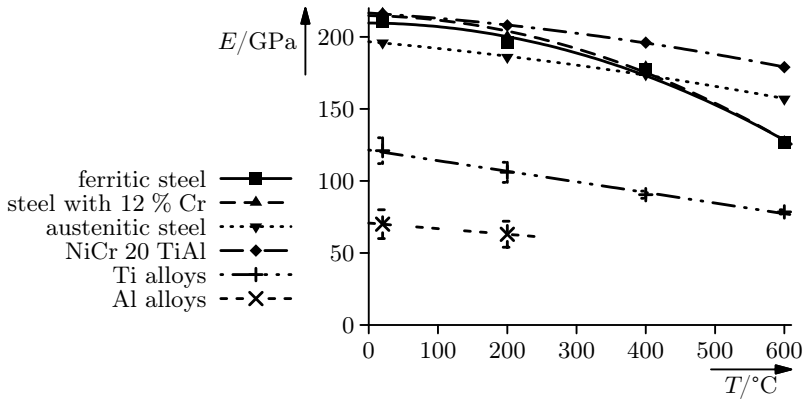


Fig. 2.15. Temperature dependence of Young’s modulus for some metals [18]

2.6 Temperature dependence of Young’s modulus

In this section, we will discuss the temperature dependence of Young’s modulus in metals and ceramics; polymer elasticity will be dealt with in chapter 8.

At typical service temperatures, which are usually smaller than half the melting temperature T_m measured in kelvin ($T < 0.5 T_m$, $[T] = \text{K}$), some rules-of-thumb can be stated for the temperature dependence of Young’s modulus. In metals, the temperature dependence of Young’s modulus E_M is rather large:

$$E_M(T) \approx E_M(0\text{K}) \cdot \left(1 - 0.5 \frac{T}{T_m}\right). \tag{2.43}$$

Here, $E_M(0\text{K})$ is Young’s modulus at 0K . Some experimentally determined values are shown in figure 2.15. The temperature dependence of Young’s modulus of ceramics is smaller [51]:

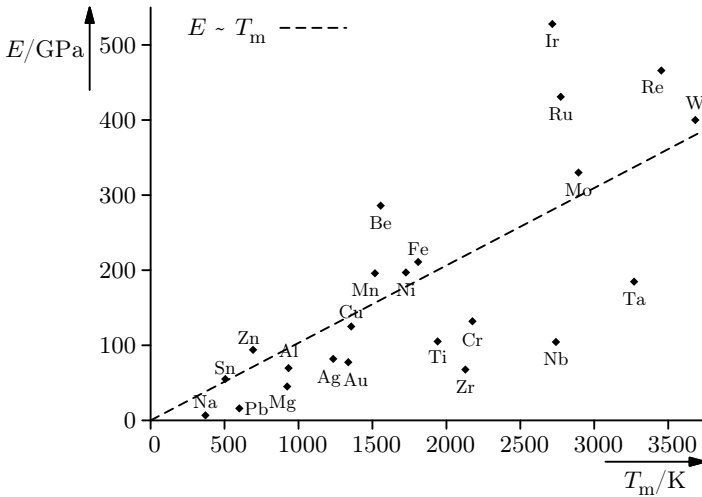


Fig. 2.16. Young's modulus versus melting temperature in some metals [18,143]

$$E_K(T) \approx E_K(0\text{K}) \cdot \left(1 - 0.3 \frac{T}{T_m}\right). \tag{2.44}$$

We can understand the temperature dependence from the properties of the atomic bond discussed in section 2.3. Raising the temperature increases the energy of the atoms by a temperature-dependent amount U_{th} . The atoms start to oscillate around their equilibrium position. The amplitude of the oscillations can be estimated by adding the thermal energy to the energy in the minimum of the potential well as sketched in figure 2.14. Because the repulsive interaction is shorter-ranged than the attractive interaction, the slope is larger on the left side of the well. The mean distance of the atoms thus grows when the temperature is raised. This explains the phenomenon of *thermal expansion*.

Due to thermal expansion, the mean equilibrium position of the atom is at a position in the potential well where the slope of the force curve and thus the stiffness is smaller – Young's modulus is reduced.

This simple model relates thermal expansion and the reduction of the elastic modulus with increasing temperature. It is confirmed by the fact that metals have a larger temperature dependence of Young's modulus than ceramics and also a larger coefficient of thermal expansion.

The reason for this is the larger bond length of the metallic bond. Because it is based on electrons in a widely spread electron gas, the interaction energy does not decrease as strongly with increasing distance as in a covalent bond that involves only two atoms. The range of the ionic bond is also rather small because the electric field is shielded by the neighbouring ions of different charges.

As a rule-of-thumb, we can state that within each class of materials, Young's modulus is roughly proportional to the melting temperature:

$$E \sim T_m. \quad (2.45)$$

This relation can also be explained with the help of figure 2.14. The energy needed to melt the material is roughly proportional to the depth of the potential well because the bonds have to be sufficiently dissolved to allow free movement of the atoms. The deeper the potential well is, the steeper are its sides, for the range of the attractive and repulsive forces are roughly the same for all materials within a certain class. As the second derivative of the energy determines the elastic properties, materials with a larger bond energy have to have a larger elastic modulus. In figure 2.16, the relation between melting temperature and Young's modulus is sketched.