P740.HW2.1.sol.tex

Supplement to: **P740.HW2.sol.tex.**

Mostly about judicious treatment of integrals.

Example 1.

Find $< y >$ defined by

$$< y >= \frac{\int_{-\infty}^{+\infty} \ dy \ y \ f(y)}{\int_{-\infty}^{+\infty} \ dy \ f(y)} = \frac{I_1}{I_0}, \tag{1}$$

where

$$f(y) = A \ exp - B(y^2 - ay) \tag{2}$$

and A is a constant fixed by norming $f(y)$ to 1, i.e., $I_0 = 1$. A formal expression that seemingly reduces your work is

$$< y >= \frac{1}{B}\frac{d}{da} \ ln(I_0). \tag{3}$$

There is still one integral to be done. Whether you choose Eq. (1) or Eq. (3) proceed something like this. Define $\kappa^2 = B$, replace y by $x = \kappa y$, Bay by $\kappa ax = 2cx$ and $dy = dx/\kappa$. The idea is to sterilize the equation, have as few things lying around a possible. Then

$$\begin{aligned} I_0 &= \int_{-\infty}^{+\infty} \ dx \ A \ exp - (x^2 - 2cx), & (4) \\ I_1 &= \frac{1}{\kappa}\int_{-\infty}^{+\infty} \ dx \ A \ x \ exp - (x^2 - 2cx) = \frac{1}{2\kappa}\frac{d}{dc} \ I_0, & (5) \\ < y > &= \frac{1}{2\kappa}\frac{d}{dc} \ ln(I_0). & (6) \end{aligned}$$

The integrals have been deliberately arranged to suggest completing the square. For example

$$I_0 = e^{c^2}\int_{-\infty}^{+\infty} \ dx \ A \ exp - (x^2 - 2cx + c^2) = exp \ c^2 \times \int_{-\infty}^{+\infty} \ dx \ A \ exp - (x - c)^2. \tag{7}$$

The integral here is a Gaussian integral centered at $x = c$. If you shift the origin to $x = c$ the integral, which cannot be done *by hand*, is independent of c, $I_0 = exp \ c^2 \times D$, where D is a number. From Eq. (6) you want the log of I_0, string it out, $ln(I_0) = c^2 + ln(D)$, you don't need D,

$$< y >= \frac{c}{\kappa} = a. \tag{8}$$

Example 2.

Suppose that for

$$f(y) = A\ exp - By^2, \tag{9}$$

where A is a constant fixed by norming $f(y)$ to 1, you want the probability that $y > b$, i.e.,

$$P(y > b) = \frac{\int_b^{+\infty} dy\ f(y)}{\int_{-\infty}^{+\infty} dy\ f(y)}. \tag{10}$$

Use x as above. Because you have the ratio of two integrals with same integrand and differential (dy) you can drop A, replace dy by dx and dress up the limit at b, $b \to \kappa b = q$,

$$P(y > b) = \frac{\int_q^{+\infty} dx\ g(x)}{\int_{-\infty}^{+\infty} dx\ g(x)} = \frac{J(q)}{2J(0)}, \tag{11}$$

where

$$g(x) = exp - x^2. \tag{12}$$

The integral $J(0)$ is $\sqrt{\pi}/2$. The error function, $erf(z)$, is defined to be

$$erf(z) = \frac{2}{\sqrt{\pi}} \int_0^z dx\ exp - x^2, \tag{13}$$

$erf(+\infty) = 1$. Thus

$$P(y > b) = \frac{1}{2}(1 - erf(q)). \tag{14}$$

$erf(x)$ is known to MATLAB.