Final takehome exam, Fluid Dynamics

Deivid Pugal

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Summary

The First problem - "An Artery for Bubba?"

Short description of the problem: Finding the velocity of blood flow in arteries, considering viscosity, fatty acid depostis or both. Also find the time to clot the artery and the rate of the "shrinkage".

The results of calculations, manipulations of the problem are listed below:

- Solutions of Eq (1) from the Exam description (case $\tau \to \infty$).

$$v_x(y) = \frac{R_0^2}{2D_\eta \rho_0} \frac{\Delta P}{L} \left(1 - \frac{y^2}{R_0^2}\right) \tag{1}$$

$$Q = \frac{2}{3} \frac{b R_0^3 \Delta P}{D_\eta L} \tag{2}$$

$$R_A^\eta = \frac{3}{2} \frac{D_\eta L}{b R_0^3} \tag{3}$$

  See details in the list item no. 1 on page 5.

- Solutions of Eq (1) from the Exam description (case $D_\eta = 0$)

$$v_x = \frac{\tau \Delta P}{\rho_0 L} \tag{4}$$

$$Q = 2 \frac{b \tau R_0 \Delta P}{L} \tag{5}$$

$$R_A^\tau = \frac{1}{2} \frac{L}{b \tau R_0}. \tag{6}$$

  See details in the list item no. 2 on page 6.

- A general solution of Eq (1) from the Exam description

$$v_x(y) = \frac{\Delta P}{D_\eta \rho_0 L \kappa^2} \left(1 - \frac{cosh\,(ky)}{cosh\,(kR_0)}\right) \tag{7}$$

$$Q = 2 \frac{b \Delta P}{D_\eta L \kappa^2} \left(R_0 - \frac{1}{\kappa} tanh\,(kR_0)\right) \tag{8}$$

  See description of above equations in the list item no. 3 on page 7.

$$R_A = \frac{L}{2b\tau R_0} \frac{z}{(z - tanh\,(z))} \tag{9}$$

$$f(z) = \frac{z}{(z - tanh\,(z))} \tag{10}$$

  See details in the list item no. 1 on page 8.

- Limits, such as $z \to \infty$ and $z \to 0$ and explanation about physical background of z could be found in the list items: 2 on page 8, 3 on page 8, and 4 on page 8.

- $f(z)$ as a function of z is shown in Figure 1 on page 3.
- Shear stress for generic radius R:

$$\sigma_{xy}(R) = \frac{\Delta P \; R}{L}. \tag{11}$$

  Details in the list item no. 1 on page 9.

- Plot $(\Delta P \sigma_0) / (P_0 |\sigma_{xy}(R)|)$ as function of κR is shown in Figure 2 on the next page.
- R as a function of time: $R = R_0 \sqrt{1 - \frac{2t}{R_0^2 \kappa^2 \tau_F}}$. Details could be found in the list item no. 3 on page 9.

Note: The following graphs use following values:

$$0.1cm < R_0 < 1cm$$

$$\frac{1}{\tau_A} = 0.1s^{-1}\left(exp\left(0.43 \cdot 3\right) - 1\right) = 0.26 \Rightarrow \tau_A = 3.8s,$$

$$\Delta P = 0.5 P_0 \Rightarrow \tau_F = 0.5\tau_A = 1.9s$$

$$D_\eta = 0.05\frac{cm^2}{s}, \; \tau = 5s \; \Rightarrow \kappa = 2\frac{1}{cm}$$

- Plot of R/R_0 as a function of t/τ_F is in Figure 3 on page 4.
- Q dependency on time with changing R.

$$Q(t) = 2b\frac{R_0^2 \Delta P}{2 D_\eta L}\left(\sqrt{R_0^2 - \frac{2t}{\kappa^2 \tau_F}} - \frac{1}{3R_0^2}\sqrt{\left(R_0^2 - \frac{2t}{\kappa^2 \tau_F}\right)^3}\right) \tag{12}$$

  . The details could be found in Eq. 49 on page 9.

- Plot of Q as a function of time in the form $Q(R(t))/Q(R_0)$ vs t/τ_F is in Figure 4 on page 4.
- Time estimation fo Bubba's artery to block. Basically it happens when

$$\frac{2t}{R_0^2 \kappa^2 \tau_F} = 1 \Rightarrow t = \frac{R_0^2 \kappa^2 \tau_F}{2} \approx \frac{1 \cdot 4 \cdot 2}{2} \approx 4s. \tag{13}$$

  A bit too fast ;).

- Considering the Eq. 48 on page 9, we can see that only if n_F is inversely proportional to x, the second segment tends to clot slower than the first one. Analogy to resistors in my opinion is that if the second resistor is "bigger" in sense of the conductivity (i.e. less resistivity), then the voltage drops more on the first resistor.

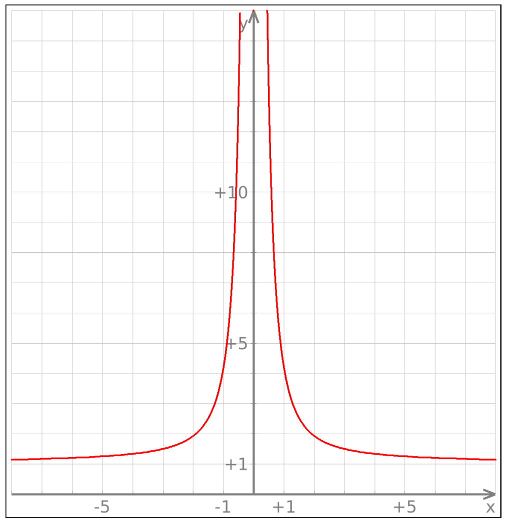

Figure 1: Grahp of $f(z) = \frac{z}{z-tanh(z)}$.

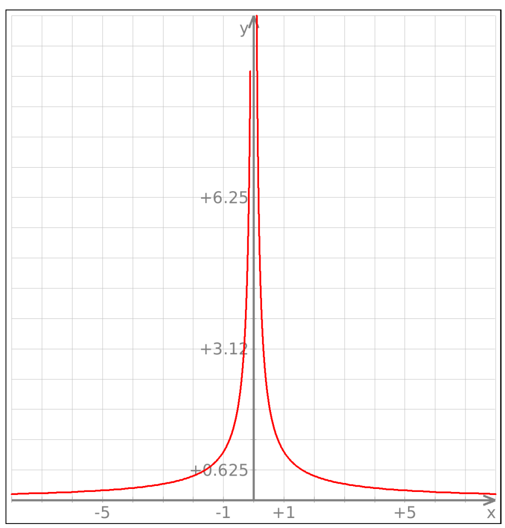

Figure 2: The plot of $(\Delta P\sigma_0) / (P_0|\sigma_{xy}(R)|)$ as a function of κR. After doing the manipulations the function takes the form $\frac{1}{\kappa R}$. At this case we considered ΔP to be a positive number. If it was negative, the graph would look like vice versa because the result would be $-\frac{1}{\kappa R}$

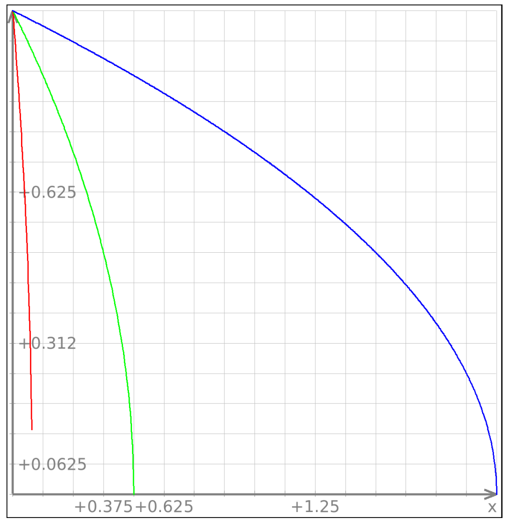

Figure 3: The Plot of R/R_0 as a function of t/τ_F. The red graph (fastest drop) represents value of $R_0 = 0.2cm$. The green graph (middle one) represents the value of $R_0 = 0.5cm$ and the blue graph $R_0 = 1cm$. The x axis is t/τ_F and the y-axis is R/R_0. (the steepest line should reach to the zero)

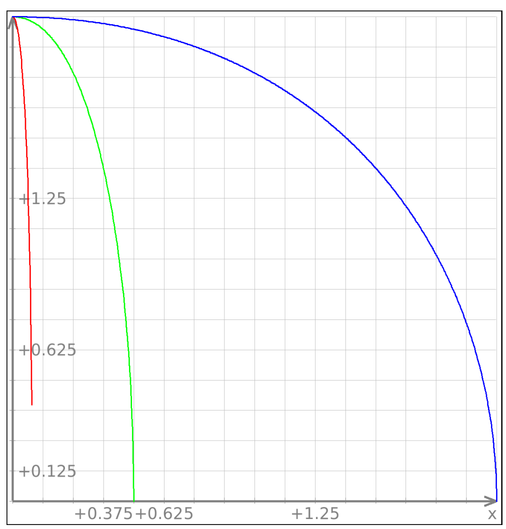

Figure 4: The Plot of $Q(R(t))/Q(R_0)$ as a function of t/τ_F. The red graph (fastest drop) represents value of $R_0 = 0.2cm$. The green graph (middle one) represents the value of $R_0 = 0.5cm$ and the blue graph $R_0 = 1cm$. The x axis is t/τ_F and the y-axis is $Q(R(t))/Q(R_0)$. (the steepest line should reach to the zero)

How to find a Submarine?

Short description of the problem: Shallow water and waves in the shallow water. A submarine in the shallow water causes disturbances in the wave frequency modes in certain cases. We'll find the location and motion of the submarine according to those disturbances.

- Normal mode frequencies of the submarine free ocean. The equation describing this kind of ocean is Jacobi differential equation and as α and β are zero in the equation, which makes it actually Legendre equation. The solution for it is a set of Legendre polynomials. See 1 on page 10.
- The shift in the normal mode frequencies due to the submarine. The shift is described by equation

$$\alpha_n = \left(\frac{d\phi_n}{dz}\right)^2_{z=p} \cdot \frac{1}{k\left(n\right)}. \tag{14}$$

See the list item no. 2 on page 10.

- One possible trajectory of the submarine in terms of z and time could be

$$\begin{aligned} & 0^{0h} \rightarrow 0.21^{2h} \rightarrow 0.65^{4h} \rightarrow 0.76^{10h} \rightarrow 0.65^{16h} \rightarrow \\ \rightarrow \quad & 0.21^{18h} \rightarrow 0^{20h} \rightarrow 0.45^{22h} \rightarrow 0.76^{26h-34h} \rightarrow 0.45^{34h}, \end{aligned}$$

  where the number shows the relative distance from the center of the shallow-water-area and the power shows at which time the submarine stays at this certain distance. For more details, see the list item no 3 on page 12.

- The color of the submarine - we could say that the color must match the water, otherwise the sattellite could take pictures of the submarine itself (at least in very shallow water). But apparently we are wrong, because it is proven long time ago that the submarine is yellow (http://en.wikipedia.org/wiki/USS_Menhaden_(SS-377) http://en.wikipedia.org/wiki/Yellow_Submarine_(song)).

Details

The First problem - "An Artery for Bubba?"

The flow of the blood in arteries is modeled as the flow between two parallel plates of separation $2R$ and width $b \gg 2R$. The flow obeys the equation of motion

$$\frac{\partial v_x}{\partial t} = -\frac{v_x}{\tau} + D_\eta \frac{\partial^2 v_x}{\partial y^2} - \frac{1}{\rho_0}\frac{\partial P}{\partial x}, \tag{15}$$

where $D_\eta = \eta/\rho_0$ and $P(x) = -(\Delta P/L)x$. We are looking for steady state solutions for the Eq (15). Also mass current will be calculated: $Q = \rho_0 b \int_{-R_0}^{+R_0} dy \cdot v_x\left(y\right)$.

1. Fixed artery radius $R_0$and $\tau \to \infty$ and $v_x(\pm R_0) = 0$. The Eq. (15) becomes
$$D_\eta \frac{\partial^2 v_x}{\partial y^2} = -\frac{1}{\rho_0}\frac{\Delta P}{L} = C. \tag{16}$$
Lets assume the solution in the form of
$$v_x = A + By^2. \tag{17}$$
So we can get the following results:
$$\begin{aligned} 2B &= -\frac{1}{D_\eta \rho_0}\frac{\Delta P}{L} = C \quad (18) \\ A &= -BR_0^2 \quad (19) \\ v_x(y) &= B\left(y^2 - R_0^2\right) \\ &= -\frac{1}{2D_\eta \rho_0}\frac{\Delta P}{L}\left(y^2 - R_0^2\right) \\ &= \frac{R_0^2}{2D_\eta \rho_0}\frac{\Delta P}{L}\left(1 - \frac{y^2}{R_0^2}\right). \quad (20) \end{aligned}$$
The mass current is:
$$\begin{aligned} Q &= b\frac{R_0^2 \Delta P}{2D_\eta L}\int_{-R_0}^{R_0}\left(1 - \frac{y^2}{R_0^2}\right)dy = \\ &= b\frac{R_0^2 \Delta P}{2D_\eta L}\left(y - \frac{1}{3}\frac{y^3}{R_0^2}\right)_{-R_0}^{R_0} = \\ &= b\frac{R_0^2 \Delta P}{2D_\eta L}\left(2R_0 - \frac{2}{3}R_0\right) = \\ &= \frac{2}{3}\frac{bR_0^3 \Delta P}{D_\eta L}. \quad (21) \end{aligned}$$
Following the analogy of electric circuit, we can find the resistance of the artery from the relation
$$Q = \frac{1}{R_A}\Delta P. \tag{22}$$
For current case:
$$R_A^\eta = \frac{3}{2}\frac{D_\eta L}{bR_0^3} \tag{23}$$

2. The limit $D_\eta = 0$ and τ is finite. This combination results in the equation
$$\frac{v_x}{\tau} = \frac{\Delta P}{\rho_0 L}, \tag{24}$$
which gives us the solution for v_x
$$v_x = \frac{\tau \Delta P}{\rho_0 L}. \tag{25}$$

For mass current we'll get

$$Q = 2\frac{b\tau R_0 \Delta P}{L}. \tag{26}$$

The resistance of the artery in this case would be

$$R_A^{\tau} = \frac{1}{2}\frac{L}{b\tau R_0}. \tag{27}$$

3. The general solution for Eq. (15). By doing some replacements (as $\kappa^2 = 1/(\tau D_\eta)$, we can get the modified form of the equation as follows

$$D_\eta \frac{\partial^2 v_x}{\partial y^2} - \frac{v_x}{\tau} = -\frac{\Delta P}{\rho_0 L}. \tag{28}$$

$$\frac{\partial^2 v_x}{\partial y^2} - \frac{1}{\tau D_\eta} v_x = -\frac{\Delta P}{D_\eta \rho_0 L}. \tag{29}$$

$$\frac{\partial^2 v_x}{\partial y^2} - \kappa^2 v_x = C. \tag{30}$$

To solve the equation, we must find a solution for the homogenous part of the equation and then a particular solution also. By adding those solution together, we'll get the general solution. First of all, the solution for the homogenous part:

$$\frac{\partial^2 v_x}{\partial y^2} - \kappa^2 v_x = 0 \tag{31}$$

$$v_x^{hom} = Ae^{ky} + Be^{-ky} \tag{32}$$

Considering the fact that we expect to see symmetric solution, we could replace the result with *cosh* function:

$$v_x^{hom} = A \cdot cosh\,(ky)\,. \tag{33}$$

It is really easy to see that a particular solution for the equation is

$$v_x^{part} = -\frac{C}{\kappa^2}. \tag{34}$$

So the general solution is

$$v_x = A \cdot cosh\,(ky) - \frac{C}{\kappa^2}. \tag{35}$$

We use boundary conditions to eliminate A. We also consider the fact that cosh function is an even function - so only R_0will be considered

$$v_x\,(y = R_0) = 0 \Rightarrow$$
$$A = \frac{C}{\kappa^2 cosh\,(kR_0)}. \tag{36}$$

So the final form of the general solution is

$$v_x = \frac{C}{\kappa^2}\left(\frac{cosh\,(ky)}{cosh\,(kR_0)} - 1\right) = \frac{\Delta P}{D_\eta \rho_0 L \kappa^2}\left(1 - \frac{cosh\,(ky)}{cosh\,(kR_0)}\right). \quad (37)$$

The resistance of the artery for the general solution is

$$\begin{aligned} Q &= \frac{b\Delta P}{D_\eta L \kappa^2}\left(y - \frac{1}{\kappa}\frac{sinh\,(ky)}{cosh\,(kR_0)}\right)_{-R_0}^{R_0} = \\ &= 2\frac{b\Delta P}{D_\eta L \kappa^2}\left(R_0 - \frac{1}{\kappa} tanh\,(kR_0)\right) \end{aligned} \quad (38)$$

By replacing $z^2 = \kappa^2 R_0^2$ and doing couple of manipulations, we'll get for Q

$$Q = \frac{2bR_0^3}{D_\eta L z^2}\left(1 - \frac{tanh\,(z)}{z}\right) \quad (39)$$

So for next we'll find various limits and resistances.

1. Now let's find the R_A in the form of $R_A = R_A^\tau f\,(z)$.

$$\begin{aligned} R_A &= \frac{D_\eta L z^3 \tau}{2bR_0^3 \tau\,(z - tanh\,(z))} = \\ &= \frac{L\kappa^2 R_0^2}{2b\tau R_0^3 \kappa^2}\frac{z}{(z - tanh\,(z))} = \\ &= \frac{L}{2b\tau R_0}\frac{z}{(z - tanh\,(z))}. \end{aligned} \quad (40)$$

   which really is in the form $R_A^\tau f\,(z)$, where $f(z) = \frac{z}{z - tanh(z)}$.

2. Let $z \to \infty$:

$$R_A \to R_A^\tau,$$

   because if $z \to \infty$, $\frac{z}{z - tanh(z)} \approx \frac{z}{z} = 1$.

3. Let $z \to 0$. In this case we can write for R_A

$$R_A = \frac{L}{2b\tau R_0 z^2}\frac{z}{\left(\frac{1}{z} - \frac{tanh(z)}{z^2}\right)} \quad (41)$$

   It's easy to check that the limit $lim_{z\to 0}\frac{z}{\left(\frac{1}{z} - \frac{tanh(z)}{z^2}\right)} = 3$. So for $R_A$we'll get

$$R_A = \frac{3}{2}\frac{LD_\eta}{bR_0^3} = R_A^\eta \quad (42)$$

4. z is the controlling physical variable, because it consists of 3 parameters, which are important in characterizing the flow. First of all it consists of the dimension of the tube. Other variables are $D_\eta$and τ. These variables show kind of characteristic time. What it means is that if we consider τ to be really big, the viscous diffusion becomes important. For smaller τ and larger $D_\eta$we'll see that fatty acid part starts to play role. Anyway, as those variables are multiplied, we can conclude that the way they influence the solution is kind of similar - the pysics behind the variables is not that different. The both variable are somehow responsible in resisting the flow of the blood. If one variable is much bigger than the other one, we could just neglect the less important variable in our inital equation. But if both are playing role, the both fit togeterh into the z very well to be a controlling physical variable.

The following section is about the reduction of the size of the artery walls.

1. Shear stress $\sigma_{xy}(R)$

$$\sigma_{xy}(R) = -\eta \frac{R_0^2 \Delta P}{2D_\eta \rho_0 L}\left(-2\frac{y}{R_0^2}\right)_{y=R} = \frac{\Delta P\ R}{L} \tag{43}$$

2. The plot could be found in Figure 2 on page 3.

3. R as a function of time, using equation $\frac{dR}{dt} = -\frac{\sigma_0}{|\sigma_{xy}(R)|}\frac{1}{\kappa}\frac{1}{\tau_A}$ and the relation $\sigma_0 = P_0/(\kappa L)$.We also consider R to be positive or zero.

$$\frac{dR}{dT} = -\frac{P_0 L}{\kappa^2 L\,|\Delta P\ R|}\frac{1}{\tau_A} = -\frac{P_0}{\kappa^2\,|\Delta P\ R|}\frac{1}{\tau_A} \tag{44}$$

$$R\ dR = -\frac{P_0 dt}{\kappa^2 \tau_A\,|\Delta P\ |} \tag{45}$$

$$\frac{1}{2}R^2 = -\frac{P_0 t}{\kappa^2 \tau_A\,|\Delta P\ |} + C \tag{46}$$

$$R = \sqrt{R_0^2 - \frac{2P_0 t}{\kappa^2 \tau_A\,|\Delta P\ |}} = \sqrt{R_0^2 - \frac{2t}{\kappa^2 \tau_F}} \tag{47}$$

$$= R_0\sqrt{1 - \frac{2t}{R_0^2 \kappa^2 \tau_F}} \tag{48}$$

4. The plot could be found in Figure 3 on page 4.

5. Q(t) would be

$$Q(t) = b\frac{R_0^2 \Delta P}{2D_\eta L}\left(y - \frac{1}{3}\frac{y^3}{R_0^2}\right)_{-R}^{R} =$$

$$= 2b\frac{R_0^2 \Delta P}{2D_\eta L}\left(\sqrt{R_0^2 - \frac{2t}{\kappa^2 \tau_F}} - \frac{1}{3R_0^2}\sqrt{\left(R_0^2 - \frac{2t}{\kappa^2 \tau_F}\right)^3}\right) \tag{49}$$

6. The plot of $Q(R(t))/Q(R_0)$ could be seen in Figure 4 on page 4. The form of the plotted function is:

$$\begin{aligned} Q(R(t))/Q(R_0) &= \frac{3}{R_0}\sqrt{R_0^2 - \frac{2t}{\kappa^2\tau_F}}\left(1 - \frac{1}{3}\left(1 - \frac{2t}{R_0^2\kappa^2\tau_F}\right)\right) \\ &= 2\sqrt{1 - \frac{2t}{R_0^2\kappa^2\tau_F}}\left(1 + \frac{t}{R_0^2\kappa^2\tau_F}\right) \end{aligned} \tag{50}$$

How to find a Submarine?

1. Normal mode frequencies of the submarine free ocean, where the depth is described by the equation $h_0(x) = h_0\left(1 - \frac{x^2}{a^2}\right)$ and waves are described by

$$\frac{\partial^2 \delta h}{\partial t^2} = g\frac{\partial}{\partial x}\left(h_0(x)\frac{\partial \delta h}{\partial x}\right). \tag{51}$$

By replacing $z = \frac{x}{a}$, $\frac{gh_0}{a^2} = \omega_0^2$, $\frac{\omega^2}{\omega_0^2} = \Omega$, we'll get the following equation:

$$\frac{\partial^2 \delta h}{\partial t^2} = \frac{g}{a^2}\frac{\partial}{\partial z}\left(h_0\left(1 - z^2\right)\frac{\partial \delta h}{\partial z}\right), \tag{52}$$

$$\frac{\partial^2 \delta h}{\partial t^2} = \omega_0^2\left(-2z\frac{\partial \delta h}{\partial z} + \left(1 - z^2\right)\frac{\partial^2 \delta h}{\partial z^2}\right) \tag{53}$$

For a steady state solution we could use the relation $\delta h(z) = H(z)\, cos(\omega t)$

$$-\omega^2 H = \omega_0^2\left(-2z\frac{dH}{dz} + \left(1 - z^2\right)\frac{d^2 H}{dz^2}\right) \tag{54}$$

$$0 = \left(1 - z^2\right)\frac{d^2 H}{dz^2} - 2z\frac{dH}{dz} + \Omega^2 H. \tag{55}$$

This equation is a special case of the Jacobi differential equation, where α and β are both zero. The Ω^2 could be expanded as $n(n+1)$. For that case the solutions for this equation are Legendre Polynomials $\phi_n(z)$. For example

$$\phi_1(z) = z \tag{56}$$

$$\phi_2(z) = \frac{1}{2}\left(3z^2 - 1\right) \tag{57}$$

2. For a submarine present at the bottom of the ocean, the depth profile looks a bit different:

$$h_0(x) = h_0\left(1 - \frac{x^2}{a^2}\right) + R^2\delta(x - b). \tag{58}$$

After replacing this into the inital equation (51) and doing similar manipulations, we'll get

$$\frac{\partial^2 \delta h}{\partial t^2} = \omega_0^2 \frac{\partial}{\partial z}\left(\left(1-z^2\right)+\frac{R^2}{h_0 a}\delta\left(z-\frac{b}{a}\right)\right)\frac{\partial \delta h}{\partial z} \tag{59}$$

Now again, after eliminating the time and by replacing $H(z)$ by

$$\theta_n(z) = \phi_n(z) + \epsilon f_n(z) + \epsilon^2 g_n(z) \tag{60}$$

and also inserting perturbation of eigenvalues

$$\nu_n^2 = \Omega_n^2 + \epsilon\alpha_n + \epsilon^2\beta_n, \tag{61}$$

we'll get the following form

$$\begin{aligned} -\left(\Omega_n^2 + \epsilon\alpha_n + \epsilon^2\beta_n\right)\left(\phi_n(z) + \epsilon f_n(z) + \epsilon^2 g_n(z)\right) &= \\ = \frac{\partial}{\partial z}\left(\left(1-z^2\right)\frac{\partial\theta_n}{\partial z}\right) + \epsilon\frac{\partial}{\partial z}\left(\delta\left(z-p\right)\frac{\partial\theta_n}{\partial z}\right) &= \qquad (62) \\ = K_0\left(z\right)\theta_n\left(z\right) + \epsilon K_1\left(z\right)\theta_n\left(z\right), & \qquad (63) \end{aligned}$$

where K_0 and $K_1$are some sort of differential operators and in the last term, the $\theta_n$could be writte explicitly using terms ϕ_n and f_n and g_n. It is easy to see that to separate all terms with one ϵ, we'll get equation

$$-\alpha_n\phi_n - \left(K_0\left(z\right) + \Omega_n^2\right) f_n = \frac{\partial}{\partial z}\left(\delta\left(z-p\right)\frac{\partial\phi_n}{\partial z}\right) \tag{64}$$

So to get a frequency shift, we have to find α_n. Lets multiply Eq. (64) by ϕ_n from left, which gives us

$$-\alpha_n\phi_n\phi_n - \phi_n K_0\left(z\right) - \phi_n\Omega_n^2 f_n = \phi_n\frac{\partial}{\partial z}\left(\delta\left(z-p\right)\frac{\partial\phi_n}{\partial z}\right) \tag{65}$$

Considering the fact that we could write for the second and third term in the equation

$$\phi_n f_n = \phi_n \sum_{n'\neq n} a_{n'}\phi_{n'} \tag{66}$$

$$K_0 f_n = \phi_n \sum_{n'\neq n} \lambda_{n'} a_{n'}\phi_{n'} \tag{67}$$

It is easy to see that the orthogonality makes the second and the third term to disappear and results $\phi_n \phi_n = 1$ in the first term, in case of integration:

$$-\int \alpha_n \phi_n \phi_n dz - \int \phi_n K_0(z)\, dz - \int \phi_n \Omega_n^2 f_n dz = \tag{68}$$

$$= \int \phi_n \frac{\partial}{\partial z}\left(\delta(z-p)\frac{\partial \phi_n}{\partial z}\right) dz \tag{69}$$

$$-\int \phi_n \left(\frac{d\phi_n}{dz}\delta'(z-p)\right) dz - \int \phi_n \frac{d^2\phi_n}{dz^2}\delta(z-p)\, dz = \alpha_n k(n) \tag{70}$$

$$\left(\frac{d\phi_n}{dz}\cdot\frac{d\phi_n}{dz} + \phi_n(p)\frac{d^2\phi_n}{dz^2} - \phi_n(p)\frac{d^2\phi_n}{dz^2}\right)_{z=p} = \alpha_n k(n) \tag{71}$$

$$\left(\frac{d\phi_n}{dz}\right)^2_{z=p} \cdot \frac{1}{k(n)} = \alpha_n, \tag{72}$$

because of the relation

$$\int f(x)\, \delta'(x-p)\, dx = -f(p)'. \tag{73}$$

$k(n)$is a norming function $\left(\frac{2n+1}{2}\right)$, because we don't have an orthonormal set of functions.

3. The submarine motion in time. At first we detect the points from the graph in Figure 6 on page 14, where are no perturbations. It is alos worth of mentioning that the modes on the graph are $n = 2...7$. These could be easily found knowing the fact that normal modes on the graph represent Ω^2 in Eq. (55) and $\Omega^2 = n(n+1)$. At those points Eq. (72) is zero, therefore the perturbation is zero. What it means that the submarine stays at the nodepoint of the wave of that frequency, causing no disturbance in wave. Considering that, I found zeroes for each node:

$$\begin{aligned}
n = 2 \quad &, \quad t = 30h\text{? almost zero..} \\
n = 3 \quad &, \quad t = 22h,\ 38h \\
n = 4 \quad &, \quad t = 4h,\ 16h \\
n = 5 \quad &, \quad t = 10h,\ 26-34h \\
n = 6 \quad &, \quad t = 0h,\ 20h \\
n = 7 \quad &, \quad t = 2h,\ 18h
\end{aligned}$$

Using this data and extermum points of Legendre polynomials ($\frac{d\phi_n}{dz}$ is zero at the extremums), we can detect the possible locations of the submarine. In Figure 5 on page 14 we can see the maximums and minimums. For nodes, they are approximately:

$$\begin{aligned} n=2 &\quad , \quad z=0 \\ n=3 &\quad , \quad z=0.45 \\ n=4 &\quad , \quad z=0,\ z=0.65 \\ n=5 &\quad , \quad z=0.28,\ z=0.76 \\ n=6 &\quad , \quad z=0,\ z=0.47,\ z=0.83 \\ n=7 &\quad , \quad z=0.21,\ z=0.59,\ z=0.87 \end{aligned}$$

Now w could estimate all kind of imaginary trajectories the submarine could have had during this 40 hour measuring cycle. Let's propose one possible movement. At $t = 0h$, the submarine was at $n = 6$, therefore it must have been either at $z = 0,\ z = 0.47$, or at $z = 0.83$. At $t = 2h$, following the same logic, the submarine must have been either at $z = 0.21,\ z = 0.59$ or at $z = 0.87$. For $t = 4h$, the shipe were at $z = 0$ or $z = 0.65$. For $t = 10h$, the submarine was at $z = 0.28$ or $z = 0.76$. For $z = 16h$, the submarine was again at 0 or $z = 0.65$. and etc. We could arrange the possible locations and the known times as follows:

$$\begin{aligned} t=0h &\quad , \quad z=0,\ 0.47,\ 0.83 \\ t=2h &\quad , \quad z=0.21,\ 0.59,\ 0.87 \\ t=4h &\quad , \quad z=0,\ 0.65 \\ t=10h &\quad , \quad z=0.28,\ 0.76 \\ t=16h &\quad , \quad z=0,\ 0.65 \\ t=18h &\quad , \quad z=0.21,\ 0.59,\ 0.87 \\ t=20h &\quad , \quad z=0,\ 0.47,\ 0.83 \\ t=22h &\quad , \quad z=0.45 \\ t=26-34h &\quad , \quad z=0.28,\ 0.76 \\ t=38h &\quad , \quad z=0.45 \end{aligned}$$

So one possible, considering that the covered distance during equal timeperiods is possibly the same, could be:

$$\begin{aligned} & 0^{0h} \rightarrow 0.21^{2h} \rightarrow 0.65^{4h} \rightarrow 0.76^{10h} \rightarrow 0.65^{16h} \rightarrow \\ \rightarrow \quad & 0.21^{18h} \rightarrow 0^{20h} \rightarrow 0.45^{22h} \rightarrow 0.76^{26h-34h} \rightarrow 0.45^{34h}. \end{aligned}$$

So the submarine goes back and forth and then moves to a new location and hangs around about 8 hours (the crew needs to rest :)) and then moves again...

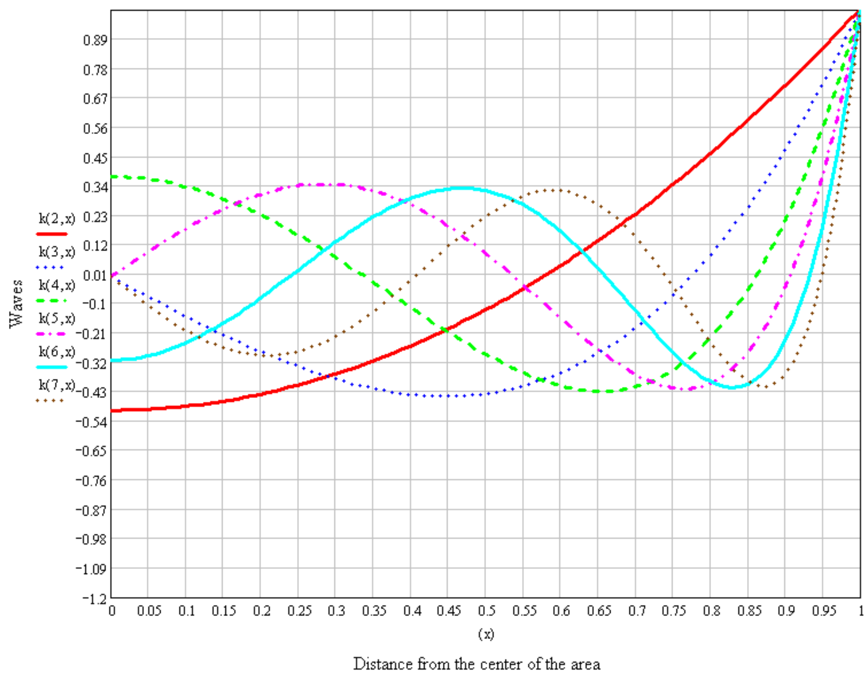

Figure 5: The Legendre polynomials, from n=2 to n=7.

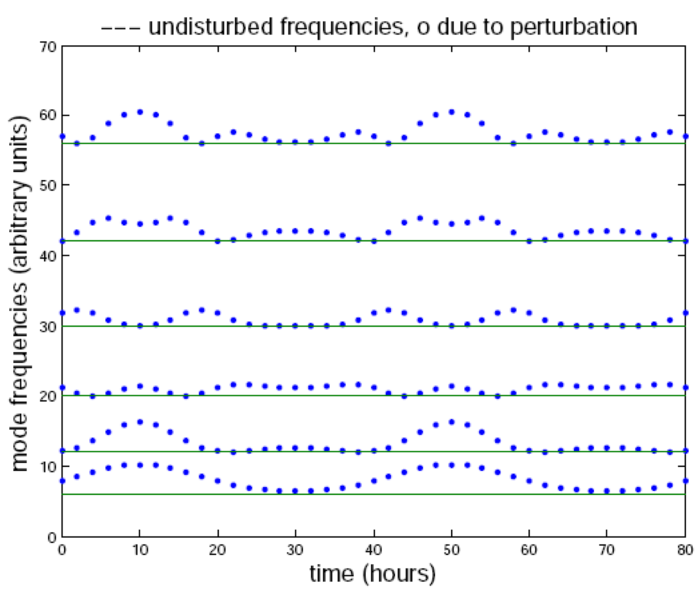

Figure 6: The mode frequencies.