HW 2, Fluid Dynamics

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0.1 Problem #1

0.1.1 Calculation of kinetic viscosity

A kinetic viscosity can be found from the relation: $D_\eta = \frac{\eta}{\rho}$. We can find the ρ from the ideal gas law as following:

$$pV = \frac{m}{M}RT \to \frac{p}{\rho} = \frac{RT}{M} \to \rho = \frac{Mp}{RT}$$

where the $p = 1atm = 101.325kPa$, $T = 300K$, $R = 8.31\frac{J}{K \cdot mol}$. So the final relation for finding the kinetic viscosity is

$$D_\eta = \frac{\eta \cdot R \ \cdot T}{p \cdot M}.$$

0.1.2 Calculation of mean free path

As derived also in class, the mean free path of a particle equals

$$l = \frac{\eta}{nmv} = \frac{\rho D_\eta}{\rho v} = \frac{D_\eta}{v},$$

where the v is an average velocity of a particle and can be found from molecular kinetic theory. As an average kinetic energy is related to the average speed, we can write:

$$\frac{1}{2}mv^2 = \frac{3}{2}kT \to v = \sqrt{\frac{3kT}{m}} = \sqrt{\frac{3RT}{M}}.$$

And the mean free path is:

$$l = \frac{D_\eta}{\sqrt{\frac{3RT}{M}}} = \frac{\eta RT}{p\sqrt{3RTM}} = \frac{\eta}{p}\sqrt{\frac{RT}{3M}}.$$

0.1.3 Calculation of cross section

The cross section of a particle is related to the mean free path as follows

$$l = \frac{1}{n \cdot \sigma}.$$

Now again we can find the n from ideal gas law:

$$pV = nRT \to pV = nkN_AT = NkT \to \frac{N}{V} = n = \frac{p}{kT}$$

and the final relation is:

$$\sigma = \frac{1}{l \cdot n} = \frac{kT}{l \cdot p} = \frac{kT\sqrt{\frac{3RT}{M}}}{D\eta p} = \frac{M\sqrt{\frac{3RT}{M}}}{\eta N_A} = \frac{\sqrt{3MRT}}{\eta N_A}$$

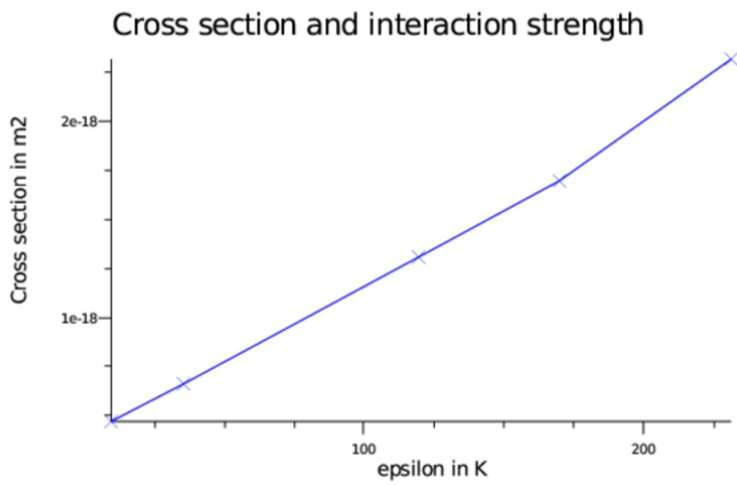

Figure 1: Cross section dependence on interaction strength

0.1.4 Calculations...

- He: $D_\eta = 1.19 \cdot 10^{-4}\frac{m^2}{s}$, $l = 8.73 \cdot 10^{-8}m$ and finally, $\sigma = 4.7 \cdot 10^{-19}m^2$
- Ne: $D_\eta = 0.381 \cdot 10^{-4}\frac{m^2}{s}$, $l = 6.24 \cdot 10^{-8}m$ and $\sigma = 6.55 \cdot 10^{-19}m^2$
- Ar: $D_\eta = 0.136\frac{\cdot 10^{-4}m^2}{s}$, $l = \pi \cdot 10^{-8}m$ and $\sigma = 1.3 \cdot 10^{-18}m^2$
- Kr: $D_\eta = 0.0723 \cdot 10^{-4}\frac{m^2}{s}$, $l = 2.42 \cdot 10^{-8}$ and $\sigma = 1.69 \cdot 10^{-18}m^2$
- Xe: $D_\eta = 0.0423 \cdot 10^{-4}\frac{m^2}{s}$, $l = 1.77 \cdot 10^{-8}m$ and $\sigma = 2.31 \cdot 10^{-18}m^2$

The requested graphs is shown in Figure 1

0.2 Problem #2

0.2.1 Relate the P and the Q

As $P =< p_x >= \frac{1}{\beta}\frac{\partial}{\partial Q}lnI_Q$ and we know that $I_Q = \int dp \cdot exp\left(-\beta\left(\frac{p^2}{2m} - Qp\right)\right)$, so to get P, we have to solve the integral for I_Q. As all values of p must pe considered, thid indefinite integral becomes a definite one and we can use table of integrals to solve the problem. It is known that the integral $\int_{-\infty}^{\infty} e^{-ax^2}e^{bx}dx = \sqrt{\frac{\pi}{a}}e^{\frac{b^2}{4a}}$. So for our case, the integral becomes

$$I_Q = \sqrt{\frac{2\pi m}{\beta}} \cdot exp\left(\frac{m\beta Q^2}{2}\right).$$

So the P would be

$$P = \frac{1}{\beta}\frac{\partial}{\partial Q} ln(I_Q) = \frac{1}{\beta}\frac{Qm\beta}{\sqrt{\frac{2\pi m}{\beta}} exp\left(\frac{m\beta Q^2}{2}\right)} exp\left(\frac{m\beta Q^2}{2}\right) = mQ$$

0.2.2 Show the relations to be equally good for finding P!=0

The first relation is:

$$f_0 \propto exp\left(-\beta\left(\frac{p^2}{2m} - pQ\right)\right), \tag{1}$$

and the second one is:

$$f_0 \propto exp\left(-\frac{\beta}{2m}(v-u)^2\right). \tag{2}$$

What we can do, is modify those equations to show that they are internally quite similar and really, P is not 0. I start with the equation (1):

$$\begin{aligned} f_0 &\propto exp\left(-\beta\left(\frac{p^2}{2m} - pQ\right)\right) = \\ &= exp\left(-\frac{\beta}{2m}\left(p^2 - 2mpQ\right)\right) = \\ &= exp\left(-\frac{\beta}{2m}\left(p^2 - 2pP\right)\right). \end{aligned} \tag{3}$$

For the equation (2), we can write down:

$$\begin{aligned} f_0 &\propto exp\left(-\frac{\beta}{2m}(v-u)^2\right) \\ &= exp\left(-\frac{\beta}{2m}u^2\right) exp\left(-\frac{\beta}{2m}\left(v^2 - 2vu\right)\right) \end{aligned} \tag{4}$$

As we can see, the equations (3) and (4) are really similar in structure. The equation (4) has an additional exponent, but this is basically constant and it does not influense the integration result in general. Both equations include a mean value. the equation (3) includes value P and the equation (4) includes u. These are constants. So, for proving that the P is not equal to zero, we have to find average of either p or v (as the v is related to p with a constant). For both cases, we'd have a integral in the form of

$$\int_{-\infty}^{\infty} x \cdot exp(-ax^2 + bx)dx.$$

And this kindof integral does not equal to zero and it is really easy to see, that symmetry of the integral is broken by term bx. That's how I've shown that both of previously described integrals are equally good for showing that $P \neq 0$.

0.2.3 Relate P to u

As u is an average velocity and P is an average momentum in a certain direction. So at this point of understading, I would say that P is just u times mass of a particle. Or at least the u is proportional to P.

0.3 Finding average of |v|

First, I'd do the normalization procedure on the equation:

$$\begin{aligned} 1 &= A\int_{-\infty}^{\infty} exp\left(-\frac{\beta}{2m}v^2\right), \\ 1 &= A\sqrt{\frac{\pi 2m}{\beta}}, \\ A &= \sqrt{\frac{\beta}{2\pi m}}. \end{aligned}$$

0.3.1 Find $|\bar{v}|$

We can find the average value of |v| as follows

$$\begin{aligned} |\bar{v}| &= \sqrt{\frac{\beta}{2\pi m}}\int d|v|\cdot|v|exp\left(-\frac{\beta}{2m}|v|^2\right) \\ &= \sqrt{\frac{\beta}{2\pi m}}\left(-\int_{-\infty}^{0} dv\cdot v\cdot exp\left(-\frac{\beta}{2m}v^2\right)+\int_{0}^{\infty} dv\cdot v\cdot exp\left(-\frac{\beta}{2m}v^2\right)\right) \\ &= \sqrt{\frac{\beta}{2\pi m}}\left(\frac{m}{\beta}exp\left(-\frac{\beta}{2m}v^2\right)\Bigg|_{-\infty}^{0}-\frac{m}{\beta}exp\left(-\frac{\beta}{2m}v^2\right)\Bigg|_{0}^{\infty}\right) \\ &= \sqrt{\frac{\beta}{2\pi m}}\frac{2m}{\beta} \\ &= \sqrt{\frac{2m}{\pi\beta}}. \end{aligned}$$

0.3.2 Probability of finding a particle faster than average.

To find the probabilty, we have to integrate from the value, found in the previous subsection, to the infinity. The integral would be:

$$P_{>} = \sqrt{\frac{\beta}{2\pi m}}\int_{\sqrt{\frac{2m}{\pi\beta}}}^{\infty} exp\left(-\frac{\beta}{2m}v^2\right).$$

An analytical result for the function is pretty nasty - including an Erf function, which is basically a power series of x - but still, it seems that it could be used to find the probability

$$
\begin{aligned}
P_> &= \sqrt{\frac{\beta\pi}{2\pi m}} \left. \frac{Erf\left(\sqrt{\frac{\beta}{2m}}v\right)}{2\sqrt{\frac{\beta}{2m}}} \right|_{\sqrt{\frac{2m}{\pi\beta}}}^{\infty} \\
&= \frac{1}{2}\sqrt{\frac{1}{2}}\left(1 - Erf\left(\sqrt{\frac{1}{\pi}}\right)\right) \\
&= \sqrt{\frac{1}{8}}\,(1 - 0.347) \\
&= 0.653\sqrt{\frac{1}{8}} = 0.231
\end{aligned}
$$

So, by my calculations, the probability would be about 25%

0.4 Problem #4

As it is a matlab problem, the program code is in a separate file. Few comments anyway. As this time wasn't said that the speed of the calculation is absolutely crucial, I generate couple of 3D plots to show variable's dependence on the time and ϵ. As it is easy to see from the result, there is a linear dependence on ϵ for the X. There is no dependence on ϵ for $<(x- <x>)^2>$.