**Physics 740: Exam 2: Solutions** 05/07/07

**Problem 1. An Artery for Bubba?** The basic equation is

$$\frac{\partial v_x}{\partial t} = -\frac{v_x}{\tau} + D_\eta \frac{\partial^2 v_x}{\partial y^2} - \frac{1}{\rho_0}\frac{\partial P}{\partial x}. \tag{1}$$

In steady state the solution to this equation is

$$v_x(y) = \frac{\tau}{\rho_0}\frac{\Delta P}{L}\left(1 - \frac{\cosh \kappa y}{\cosh \kappa R_0}\right), \tag{2}$$

where $\kappa^2 = 1/(\tau D_\eta)$. The total mass current is

$$Q = A\tau\frac{\Delta P}{L}\left(1 - \frac{\tanh \kappa R_0}{\kappa R_0}\right), \tag{3}$$

where $A = 2R_0 b$.

1. For $D_\eta \to 0$, $\kappa \to \infty$ and
$$Q = A\tau\frac{\Delta P}{L} = \frac{1}{R_A^\tau}\Delta P. \tag{4}$$

2. For $\tau \to \infty$, $\kappa \to 0$, $tanhz/z \to (1 - z^2/3)$ and
$$Q = A\frac{R_0^2}{3D_\eta}\frac{\Delta P}{L} = \frac{1}{R_A^\eta}\Delta P. \tag{5}$$

3. These answers can be put in *ohms law* form
$$Q = \frac{1}{R_A(z)}\Delta P, \tag{6}$$
where $R_A(z) = R_A^\tau f(z)$ and $f(z) = 1/(1 - \tanh z/z)$.

4. The controlling variable $z^2 = \kappa^2 R_0^2$ is the ratio of two times, $\tau$ and $R_0^2/D_\eta$. The former is the time for momentum loss pointwise in the artery and the latter is the time for momentum loss by diffusion to the artery walls.

The shear stress at the artery wall is

$$|\sigma_{xy}(R)| = \left| -\eta\left(\frac{\partial v_x}{\partial y}\right)_{y=R} \right| = \frac{\Delta P}{\kappa L}\tanh \kappa R. \tag{7}$$

Thus the equation for the rate of change of the artery radius is

$$\frac{dz}{dt} = \frac{1}{\tanh z}\frac{1}{\tau_F}, \tag{8}$$

where $\tau_F = \tau_A \Delta P/P_0$. This equation has solution

$$z = cosh^{-1}\left[e^{-\tau}\cosh z_0\right], \quad \tau = t/\tau_F. \tag{9}$$

Note $R = 0$ or $z = 0$ corresponds to $e^{-\tau}\cosh z_0 = 1$. So the *lifetime* of the artery is

$$\tau = ln(\cosh z_0) \rightarrow z_0, \tag{10}$$

where the formula on the right is for $z_0 \gg 1$.

**Oops!** The formula for $\tau_A$ was typed incorrectly. It should be

$$\frac{1}{\tau_A} = \frac{\kappa a}{\tau_A(0)}\left(\frac{n_F}{n_F^c} - 1\right), \tag{11}$$

With the numbers given and the corrected formula the lifetime of the artery is of order one year, $10^7$ sec. With the uncorrected formula it is a few seconds.

For two identical resistors in series, with a constant voltage across them, an increase in one of the resistors causes the voltage difference to shift with the larger fraction of the voltage difference across the larger resistor; equivalently, a smaller voltage difference across the smaller resistor. Transcribe to the problem at hand. As part of an artery clogs it causes pressure gradient reduction in the unclogged parts of the artery. This pressure reduction causes an increase in the rate of clogging of the unclogged parts of the artery. The system is not forgiving!

**Problem 2. How to find a Submarine?** In scaled form the equation for the height fluctuations of the ocean is

$$\frac{\partial^2 \delta h}{\partial \tau^2} = \frac{\partial}{\partial z}\left(Q(z)\frac{\partial \delta h}{\partial z}\right), \tag{12}$$

where

$$Q(z) = \left(1 - z^2\right) + \epsilon\, \delta(z-p), \quad -1 < z < 1, \tag{13}$$

$z = x/a$, $\tau = tc_0/a$, $c_0^2 = gh_0$ and $\epsilon = R^2/ah_0$.

1. The unperturbed problem starts with $\delta h = H(z)\, exp{-i\Omega\tau}$ ($\Omega^2 = \omega^2/\omega_0^2$, $\omega_0^2 = gh_0/a^2$)

2. The equation for $H$ is Legendre's equation

$$(1-z^2)\frac{d^2H}{dz^2} - 2z\frac{dH}{dz} + \Omega^2 H = 0 \tag{14}$$

so the eigenfrequencies of the ocean are

$$\Omega^2 = n(n+1) \tag{15}$$

and associated with the orthonormal set $\{P_n(z)\}$, the Legendre polynomials.

3. When the frequency shift due to the perturbation is found in lowest order perturbation theory the result is

$$\delta\Omega_n^2 = \frac{2n+1}{2}\epsilon\left(\frac{dP_n(z)}{dz}\right)^2_{z=p} \tag{16}$$

or

$$\Omega_n^2 = n(n+1) + \frac{2n+1}{2}\epsilon\left(\frac{dP_n(z)}{dz}\right)^2_{z=p}. \tag{17}$$

The submarine does not disturb the frequency of a normal mode of the ocean if it is sitting on an anti-node of the mode.

1. From the numerical data in Fig. 3 at times [0,2,4,10,16,18,20,22,26,28,30,32,34] the submarine is on an anti-node of modes [6,7,4,5,4,7,6,3,5,5,5,5,5]. The motion is periodic so I stopped after 35 hours.

2. Find the location of the anti-nodes of the Legendre polynomials, $n = 2\cdots 7$. There is no magic here. You just have to deal with numbers.

3. Follow the submarine.

4. The result in the figure here is from the simplest anaylsis, The horizontal lines are the anti-node positions.

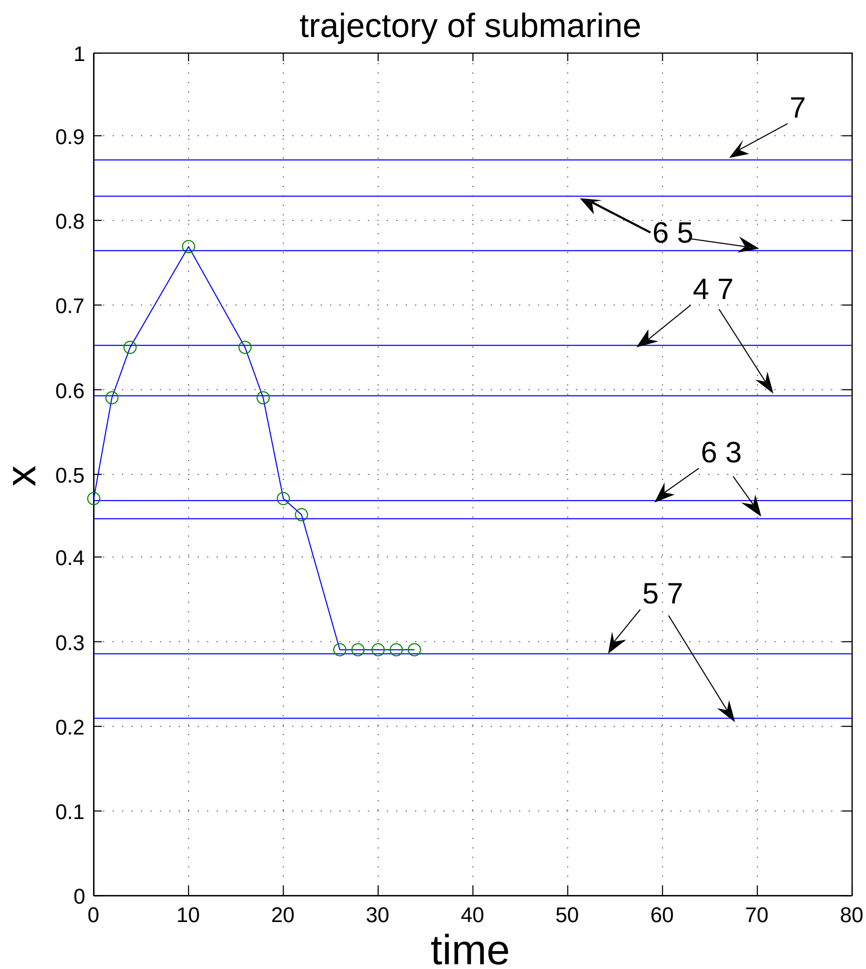


FIG. 1: Position of the submarine as a function of time. Horizontal lines are anti-node locations.