Physics 740: Spring 2006:

Afternote.2.tex 02/07/07

Note on the Boltzmann Equation and Fluid Dynamics.

A. The equations of fluid dynamics (somewhat specialized):

1. conservation of mass
$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0. \tag{1}$$

2. conservation of momentum
$$\left(\frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla \right) \mathbf{u} = \frac{\mathbf{F}}{m} - \frac{1}{\rho} \nabla P + \frac{D_\eta}{3} \nabla (\nabla \cdot \mathbf{u}) + D_\eta \nabla^2 \mathbf{u}. \tag{2}$$

3. conservation of energy
$$\left(\frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla \right) \theta = -\frac{2}{3} (\nabla \cdot \mathbf{u}) \theta + \frac{2}{3} \frac{\kappa}{\rho} \nabla^2 \theta. \tag{3}$$

B. How to get to the equations of fluid dynamics from the Boltzmann equation: Overview.

1. Write the Boltzmann equation for the $(\mathbf{x}, \mathbf{p})$ distribution function, $f(\mathbf{x}, \mathbf{p}, t)$, (in the relaxation time approximation to the collision process),
$$Lf = \left(\frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla_{\mathbf{x}} + \mathbf{F} \cdot \nabla_{\mathbf{p}} \right) f = -\frac{f - f_0}{\tau}, \tag{4}$$
where f_0 is the equilibrium distribution function and L the Boltzmann operator.

2. From the Boltzmann equation construct an equation for the average of a quantity, A, conserved in collision.
$$\frac{\partial < nA >}{\partial t} + \nabla_{\mathbf{x}} \cdot < nA\mathbf{v} > - < n\mathbf{v} \cdot \nabla_{\mathbf{x}} A > - < n\mathbf{F} \cdot \nabla_{\mathbf{p}} A >= 0, \tag{5}$$
where
$$< A(\mathbf{x}, t) >= \frac{1}{n} \int d\mathbf{p} \; A \; f(\mathbf{x}, \mathbf{p}, t). \tag{6}$$

3. Work out the 3 conservation laws (for $A = m$, $A = mv_i$ and $A = m(\mathbf{v} - \mathbf{u})^2/2$) in general form. These equations, involving quantities defined in terms of averages over the f, are

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0, \tag{7}$$

$$\left(\frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla \right) u_i = \frac{1}{m} F_i - \sum_k \frac{1}{\rho} \frac{\partial P_{ik}}{\partial x_k}, \tag{8}$$

$$\rho \left(\frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla \right) \theta = -\frac{2}{3} \nabla \cdot \mathbf{Q} - \frac{2}{3} \sum_k \sum_i P_{kj} M_{jk} = 0, \tag{9}$$

The defined quantites are:

$$n(\mathbf{x}, t) = \int d\mathbf{p} \; f(\mathbf{x}, \mathbf{p}, t), \tag{10}$$

$$< \cdots (\mathbf{x}, t) > = \frac{1}{n} \int d\mathbf{p} \; \cdots \; f(\mathbf{x}, \mathbf{p}, t), \tag{11}$$

$$\rho(\mathbf{x}, t) = mn, \tag{12}$$

$$\mathbf{u}(\mathbf{x}, t) = < \mathbf{v} >, \tag{13}$$

$$\theta(\mathbf{x}, t) = \frac{1}{3} m < |\mathbf{v} - \mathbf{u}|^2 >, \tag{14}$$

$$\mathbf{Q}(\mathbf{x}, t) = \frac{1}{2} \rho < (\mathbf{v} - \mathbf{u}) |\mathbf{v} - \mathbf{u}|^2 >, \tag{15}$$

$$P_{ij}(\mathbf{x}, t) = \rho < (v_i - u_i)(v_j - u_j) >, \tag{16}$$

$$M_{ik}(\mathbf{x}, t) = \frac{m}{2} \left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i} \right). \tag{17}$$

Note the similarity between Eqs. (7)-(9) and Eqs. (1)-(3). To work out $\mathbf{u}$, $\mathbf{Q}$, etc. we need an explicit expression for f.

C. Solution for f. Write Eq. (4) in the form

$$f = \frac{1}{1 + \tau L} f_0 = f_0 - \tau L f_0 + \tau L \tau L f_0 = \cdots. \tag{18}$$

1. Find the form of the conservation laws when the averages over f are calculated using

$$f^{(0)} = f_0, \tag{19}$$

$$< A(\mathbf{x}, t) >_{(0)} = \frac{1}{n} \int d\mathbf{p} \; A \; f^{(0)}(\mathbf{x}, \mathbf{p}, t), \tag{20}$$

the *zeroth order* approximation.

2. Find the form of the conservation laws when the averages over f are calculated using

$$
\begin{aligned}
f^{(1)} &= f_0 - \tau L f_0, && (21)\\
< A(\mathbf{x},t) >_{(1)} &= \frac{1}{n}\int d\mathbf{p}\; A\; f^{(1)}(\mathbf{x},\mathbf{p},t), && (22)
\end{aligned}
$$

the *first order* approximation.

D. Averages over f. For f_0 take

$$
f_0 = \frac{n}{(2\pi m\theta)^{3/2}}\; exp - \frac{m}{2\theta}(\mathbf{v}-\mathbf{u})^2, \qquad \int d\mathbf{p}\; f_0 = n, \tag{23}
$$

where $\mathbf{p} = m\mathbf{v}$ and n, $\mathbf{u}$ and θ are functions of $(\mathbf{x},t)$.

C.1. Zeroth order approximation.

C.1.1 Averages.

$$
\begin{aligned}
Q^{(0)} &= 0, && (24)\\
P^{(0)}_{ij} &= \delta_{ij}\theta, && (25)
\end{aligned}
$$

C.1.2 Conservation Laws.

$$
\begin{aligned}
\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{u}) &= 0, && (26)\\
\left(\frac{\partial}{\partial t} + \mathbf{u}\cdot\nabla\right)\mathbf{u} &= \frac{\mathbf{F}}{m} - \frac{1}{\rho}\nabla P, \quad (Euler) && (27)\\
\left(\frac{\partial}{\partial t} + \mathbf{u}\cdot\nabla\right)\theta &= -\frac{2}{3}(\nabla\cdot\mathbf{u})\theta. && (28)
\end{aligned}
$$

C.2. First order approximation.

C.2.1 Averages.

$$
\begin{aligned}
Q^{(1)} &= -\kappa\nabla\theta, && (29)\\
P^{(1)}_{ij} &= P^{(0)}_{ij} - 2\frac{\eta}{m}M_{ij} + \frac{2}{3}\eta\delta_{ij}\nabla\cdot\mathbf{u}, && (30)
\end{aligned}
$$

where there are equations for the transport coefficients η and κ,

$$
\begin{aligned}
\kappa &= \frac{m^2\tau}{6\theta}\int d\mathbf{p}\; w^4\left(\frac{m}{2\theta}w^2 - \frac{5}{2}\right) f_0 = \frac{5}{2}n\theta\tau, && (31)\\
\eta &= \frac{m^2\tau}{\theta}\int d\mathbf{p}\; w_x^2 w_y^2\; f_0 = n\theta\tau. && (32)
\end{aligned}
$$

C.2.2 Conservation Laws.

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0, \tag{33}$$

$$\left(\frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla\right) \mathbf{u} = \frac{\mathbf{F}}{m} - \frac{1}{\rho} \nabla P + \frac{D_\eta}{3} \nabla (\nabla \cdot \mathbf{u}) + D_\eta \nabla^2 \mathbf{u}, \quad (Navier - Stokes) \tag{34}$$

$$\left(\frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla\right) \theta = -\frac{1}{C_V} (\nabla \cdot \mathbf{u}) \theta + D_\kappa \ \nabla^2 \theta, \tag{35}$$

where $C_V = 3/2$, $D_\eta = \eta/\rho$ and $D_\kappa = \kappa / C_V / \rho$.

E. Observations about the conservation laws, Eqs. (33)-(35).

1. There are 5 equations for 6 variables, ρ, θ, P and the 3 components of $\mathbf{u}$. These equations are **closed** by using an EOS, e.g., $P(\rho, T)$,

$$dP = \left(\frac{\partial P}{\partial \rho}\right)_T d\rho + \left(\frac{\partial P}{\partial \theta}\right)_\rho d\theta. \tag{36}$$

2. There are 2 transport coefficients, the kinematic viscosity, D_η and the thermal diffusivity, D_κ, both are diffusion constants.

   (a) Both are proportional to τ. Both scale in the same way with mean free path, temperature, density, Eqs. (31) and (32), for an ideal gas.

   (b) For $\tau \to 0$ Eqs. (33)-(35) reduce to the Euler equation, i.e., Eqs. (26)-(28). This is because as $\tau \to 0$ $f \to f_0$ very rapidly, Eq. (4) so that f never gets very far from f_0.

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3. For $\mathbf{u} = 0$ the Navier-Stokes equation reduces to the *equation of hydrostatic equilibrium*,

$$\frac{\mathbf{F}}{m} - \frac{1}{\rho} \nabla P = 0, \tag{37}$$

and the energy conservation equation reduces to the*Fourier heat law.*

$$\frac{\partial \theta}{\partial t} = D_\kappa \ \nabla^2 \theta. \tag{38}$$

4. If ρ is a constant $\nabla \cdot \mathbf{u} = 0$ and the fluid is said to be *incompressible.* This is often the case.

5. When both $\tau \to 0$ and $\nabla \cdot \mathbf{u} = 0$ the fluid is called an *ideal fluid.* The study of ideal fluids produces many useful results.