HW 4, Fluid Dynamics

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0.1 Problem #1

Some useful data for further calculations

$$
\begin{aligned}
M_S &= 2 \cdot 10^{30} kg \\
R_S &= 695 \cdot 10^{6} m \\
M_N &= 1.675 \cdot 10^{-27} kg \\
T_S &= 25 days = 2.16 \cdot 10^{6} s
\end{aligned}
$$

Where M is correspondigly mass of the Sun and a neutron. R is the radius of the sun.

0.1.1 McCarran calculation

The density of the cindery is about $n_N = 10^{45} m^{-3}$. We get the mass density $\rho_S = n_N \cdot M_N$. Total volume of the cinder will be then $V = \frac{M_S}{\rho_S} = \frac{M_S}{n_N M_N}$. From the volume we can estimate the radius of the cinder by using the volume calcuation relation for a spherical body as follows:

$$\frac{4}{3}\pi r^3 = \frac{M_S}{n_N M_N} \rightarrow r = \sqrt[3]{\frac{3M_S}{n_N M_N 4\pi}} \approx 6600m$$

Google maps gave the approximate value of the length of McCarran Blvd 35km. So the radius would be: $R_{MC} \approx 5500m$. Knowing that fact, the radious of a neutron star would be $1.2 McCarrans$.

0.1.2 Rotation of the cinder

For a rigid spherical body the moment of inertia is $I = \frac{5}{2} M_S R_S^2$ and we can find angular momentum $L = I \times \omega$ and $\omega = \frac{2\pi}{T}$. As the angular momentum of the body is constant, we can write down the equation

$$\frac{5}{2} M_S R_S^2 \frac{2\pi}{T_S} = \frac{5}{2} M_S R_N^2 \frac{2\pi}{T_N}$$

which gives us a relation

$$T_N = \frac{T_S R_N^2}{R_S^2} \approx 0.2 ms$$

So cinder makes full rotation in about 0.2ms. The Sun couldn't rotatate that fast, because of the centrifugal force which probably would blow away most of the star.

0.1.3 Pauli ja gravitational pressures

Again, we have the pressure equation for a self-gravitating body

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(\frac{r^2}{\rho}\frac{\partial P}{\partial r}\right) = -4\pi G\rho$$

As formed Pauli pressure is about known... and also the density, we can write

$$\begin{aligned} P &= \frac{N}{V}\frac{\hbar^2}{M_N^2}\left(\frac{N}{V}\right)^{\frac{2}{3}} \\ \rho &= M_n\left(\frac{N}{V}\right) \end{aligned}$$

so if we do not consider the term $\left(\frac{N}{V}\right)^{2/3}$ in calculating the derivative of $\frac{\partial P}{\partial r}$. I tried considering the term also, but things got only messy for some reason. At least I wasn't able to extract anything even close reminding the pressures. Of coure, this term also doesn't chage as much as the term of power of 1... So there even might be some physical justification for that. Following this outline, wel'll get a set of equations:

$$\frac{\partial P}{\partial r} = \frac{\hbar^2}{M_N^2}\nu^{2/3}\frac{\partial\rho}{\partial r}$$

By replacing this into the the initial equation and also considering the facts that $r = x \cdot R$ and $\rho = \frac{yM}{R^3}$, we'll get:

$$\begin{aligned} \frac{1}{r^2}\frac{\partial}{\partial r}\left(\frac{r^2}{\rho}\frac{\partial\rho}{\partial r}\right) &= \frac{-4\pi G M_N^2\rho}{\hbar^2\nu^{2/3}} \\ \frac{1}{x^2}\frac{\partial}{\partial r}\left(\frac{x^2}{\rho}\frac{\partial\rho}{\partial x}\right) &= \frac{-4\pi G M_N^2 R^2\rho}{\hbar^2\nu^{2/3}} \\ \frac{1}{x^2}\frac{\partial}{\partial r}\left(\frac{x^2}{y}\frac{\partial y}{\partial x}\right) &= \frac{-4\pi G M_N^2 R^2 yM}{\hbar^2\nu^{2/3}R^3} \\ \frac{1}{x^2}\frac{\partial}{\partial r}\left(\frac{x^2}{y}\frac{\partial y}{\partial x}\right) &= \frac{-4\pi G M_N^2 yM}{\hbar^2\nu^{2/3}R} \end{aligned}$$

If we multiply the left side with $\frac{MR^3}{R^3M}$, we'll get the term $\frac{GM^2}{R^4}$, which is the gravitational pressure (without any numeric constants). It is about $4 \cdot 10^{35}Pa$ which is a really huge number. The rest of the term is therefore Pauli pressure and it is something like that: $\frac{-4\pi M_N^2 R^3}{\hbar^2\nu^{2/3}M}$. By checking units, which are $\frac{kg^2m^3s^4m^2}{m^2m^2kg^2\cdot m^2kg\cdot s^2} = \frac{s^2}{kg\cdot m}$ which is inverse pressure. The size of this term is approximately $-2\cdot 10^{-34}$ So if to multiply those terms, we get the constant about -100.

0.1.4 Velocity of sound of neutron matter

This time, let's define the pressure - density relationship as follows:

$$P = \frac{N}{V}\frac{\hbar^2}{M_N}\left(\frac{N}{V}\right)^{2/3} = \rho^{5/3}\frac{\hbar^2}{M_N^{8/3}}.$$

The derivative is

$$\frac{\partial P}{\partial \rho} = \frac{5}{3}\rho^{2/3}\frac{\hbar^2}{M_N^{8/3}} = c^2$$

$$c = \sqrt{\frac{5}{3}}\rho^{1/3}\frac{\hbar}{M_N^{4/3}}$$

The numeric value for the neutron star would be (using values from previous subsections, where we found the density of the star)

$$c \approx 80 \cdot 10^6 \frac{m}{s}$$

So, the period of a frequency for a 10 000m diameter star would be about $125\mu s$ and the frequency about $50kHz$. For The Sun... If sun would vibrate with those frequencies, the speed of the sound must be ca. $11 \cdot 10^{12} \frac{m}{s}$ which wouldn't make Einstein happy at all... (speed of light is a few orders of magnitude lower than this result).

0.2 Vortices, calculations.

Some facts able to get from looking those graphs on Fig 1. a bit.

- Distance between the vortices is a constant
- They have common center around which the are turning.
- Turning period is the same
- Turning directions depend on signs of K1 and K2

Although, I wasn't able do get common formulas describing the movement of velocities, I still have some ideas how it could be done the simplest way.

I think that complex number description is a really good solution for those problems, because we can imagine that the plane is a complex plane. Then tehere is only needed a term $Re^{ict}$for both of the circles, where c and R are dependent on K1 and K2. Also, the center of rotation is not at the origin, so there is another term needed, which switches the center of the rotation to the correct place. It would also be dependent on K1 and K2

Another way to get solutions for this problem, is to solve the equation system:

$$\begin{aligned}
\dot{x_1} &= \frac{-k_2}{R^2}(y_1 - y_2) \\
\dot{y_1} &= \frac{k_2}{R^2}(x_1 - x_2) \\
\dot{x_2} &= \frac{k_1}{R^2}(y_1 - y_2) \\
\dot{y_2} &= \frac{-k_1}{R^2}(x_1 - x_2)
\end{aligned}$$

There are few ways how to do solve it. One is to construct a matrix of constnts k's and solve an eigenvalue problem. It should eventually give a solution. Another way is to use Laplace'i transform. I started this but I ran out of time. Anyway, Here I write how it would work generally with laplace transform. Notice that there is no R^2in the equations, because I included it into the coefficients. So transformed equations would be

$$\begin{aligned}
sX_1(s) - x_1(0) &= -k_2 Y_1(s) + k_2 Y_2(s) \\
sY_1(s) - y_1(0) &= k_2 X_1(s) - k_2 X_2(s) \\
sX_2(s) - x_2(0) &= k_1 Y_1(s) - k_1 Y_2(s) \\
sY_2(s) - y_2(0) &= -k_1 X_1(s) + k_1 X_2(s)
\end{aligned}$$

And, for instance, the relationship between $Y_1$and Y_2 would be for this case

$$Y_2 = \frac{s^2 Y_1 - s + Y_1 \left(k_2^2 + k_2 k_1\right)}{k_2^2 + k_2 k_1}$$

It wouldn't be really hard to find reverse Laplace'i transform for this kind of equations, because most terms are constant and not really complicated.

0.2.1 Rest of the second problem

Is in a separat .m file.

0.3 More about a vortex

Velocity is given as $\vec{v}(r) = \frac{\hbar}{m}\frac{1}{r}\vec{e}_\theta, r \geq a$. All calculations are reasonable to do in polar coordinates. So, step by step it would be:

$$\begin{aligned}
\vec{v} \cdot \nabla &= \frac{\hbar}{m}\frac{1}{r^2}\frac{\partial}{\partial\theta} \\
(\vec{v} \cdot \nabla)\,\vec{v} &= \frac{\hbar}{m}\frac{1}{r^2}\frac{\partial}{\partial\theta}\left(\frac{\hbar}{m}\frac{1}{r}\vec{e}_\theta\right) = \frac{\hbar^2}{m^2 r^3} \cdot (-\hat{r}) \\
-\frac{\hbar^2}{m^2 r^3} \cdot \hat{r} &= -\frac{1}{\rho_0}\left(\frac{\partial P}{\partial r}\right)\hat{r} \\
\frac{\partial P}{\partial r} &= \frac{\hbar^2 \rho_0}{m^2 r^3} \\
P &= -\frac{1}{2} \cdot \frac{\hbar^2 \rho_0}{m^2 r^2} + P_0
\end{aligned}$$

So we can found here z(r)

$$z(r) = -\frac{1}{2} \cdot \frac{\hbar^2}{m^2 r^2 g}$$

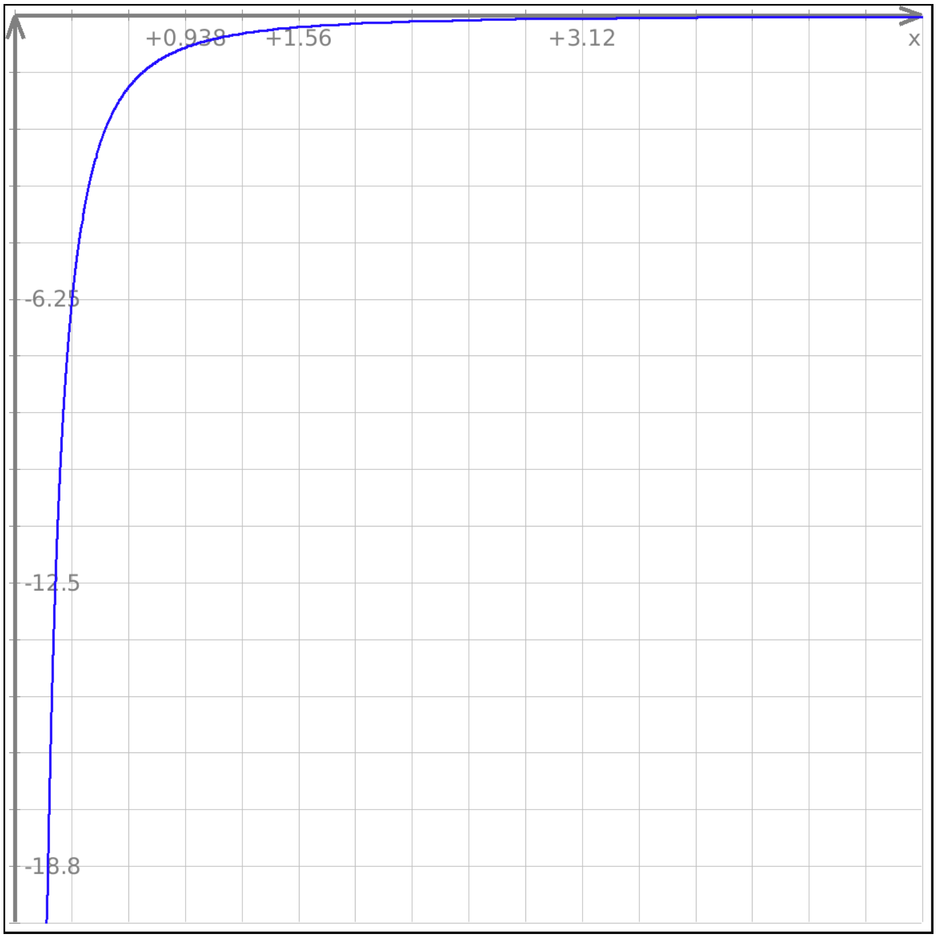

Figure 1: To fill the extra space on the last page, there is the dimple dependence of r.

The dimple for the case of He superfluid would be then

$$z(r = 2.56 \cdot 10^{-10} m) = -\frac{1}{2} \cdot \frac{(1.05 \cdot 10^{-34} J \cdot s)^2}{(4 \cdot 1.66 \cdot 10^{-27} kg)^2 (2.56 \cdot 10^{-10} m)^2 9.8 \frac{m}{s^2}} \approx -200m$$

The value came really high, for some reason. Of course the calculation is for really small dimensions.. Physically I could imagine hole at this point.. hole in the superfluid. See also Fig. 1