P740.HW7.sol.tex

1. Hwy 66 traffic.

From the numerical data $D \approx -7$ m/sec. This should be in accord with H.1,

$$D = \frac{F_R - F_L}{\rho_R - \rho_L}, \tag{1}$$

where F is the flux, $F = \rho u$, and L and R are to the left and right of the shock respectively. I used the two boxes $30 \le t \le 40$ and $60 \le x \le 100$ and $100 \le x \le 140$ on the left and the same two x intervals and $60 \le t \le 70$ on the right. I averaged the flux and density in each pair of boxes and found

$$F_L = 0.5,\ cars/sec, \tag{2}$$
$$F_R = 0.3,\ cars/sec, \tag{3}$$
$$\rho_L = 0.07,\ cars/m, \tag{4}$$
$$\rho_R = 0.11,\ cars/m. \tag{5}$$

In H.1 these numbers yield $D = -5$ m/sec. Given the noisiness of the data this is pretty OK, the right sign and about the right number.

2. Car motion.

The equation to be solved is

$$\dot{x} = 0,\ t < -x_0/v_m, \tag{6}$$
$$\dot{x} = \frac{v_m}{2}\left(1 + \frac{x}{v_m t}\right),\ t > -x_0/v_m, \tag{7}$$

where $x(t=0) = x_0 < 0$. Use $\tau = v_m t$, $\tau_0 = -x_0/v_m$, to write

$$\frac{dx}{d\tau} = \frac{1}{2}\left(1 + \frac{x}{\tau}\right). \tag{8}$$

Substitute $x = u\tau$, express dx in terms of $d\tau$ and du and re-arrange

$$\frac{du}{1-u} = \frac{1}{2}\frac{d\tau}{\tau}. \tag{9}$$

Integrate from u_0, τ_0 to u, τ with the result

$$1 - u = (1-u_0)\sqrt{\frac{\tau_0}{\tau}}, \tag{10}$$
$$u = 1 - (1-u_0)\sqrt{\frac{\tau_0}{\tau}}, \tag{11}$$
$$x = \tau - 2\sqrt{\tau_0 \tau}, \tag{12}$$

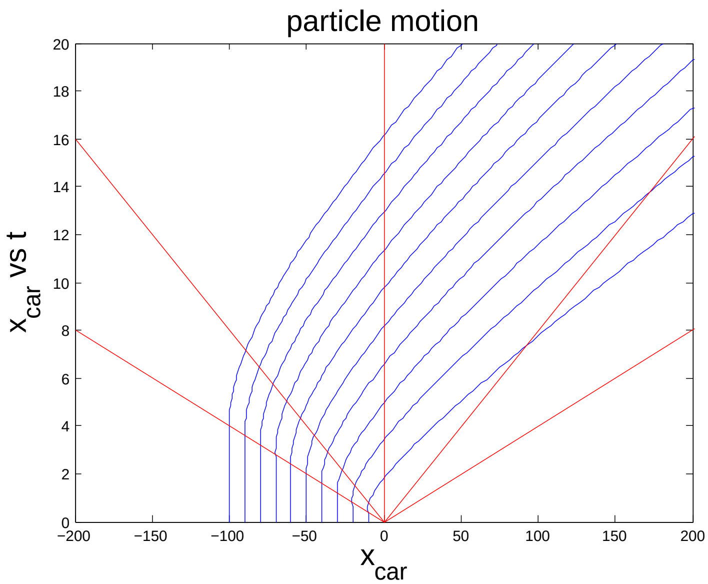

FIG. 1: Position of cars vs t.

where the last line follows from $u_0 = x_0/\tau_0 = -1$. At $\tau = \tau_0$, $x = \tau_0$, etc. For two cars at starting from $x_0(1) < x_0(2)$ the separation at time after $\tau_0(2)$ is

$$x_1(\tau) - x_2(\tau) = 2\left(\sqrt{\tau_0(2)} - \sqrt{\tau_0(1)}\right)\sqrt{\tau}, \tag{13}$$

i.e., the separation continues to grow in time.

3. Nonlinear equation.

Using the expansion of Q in powers of λ leads to the hierarchy of equations

$$\begin{aligned}
\ddot{Q}_1 &= -\alpha^2 Q_1 + Ae^{-i\omega t}, &(14)\\
\ddot{Q}_2 &= -\alpha^2 Q_2 + \gamma\alpha^2 Q_1^2 &(15)\\
\ddot{Q}_3 &= -\alpha^2 Q_3 + 2\gamma\alpha^2 Q_1 Q_2 &(16)\\
&\vdots\ . &(17)
\end{aligned}$$

These can be solved systematically, for Q_1, Q_2, etc. with the result

$$\begin{aligned}
Q_1 &= A_1 e^{-i\omega t}, &(18)\\
Q_2 &= A_2 e^{-i2\omega t} &(19)\\
Q_3 &= A_3 e^{-i3\omega t} &(20)\\
&\vdots\ . &(21)
\end{aligned}$$

with $A_1 = A/(\alpha^2 - \omega^2)$, $A_2 = A_1^2/(\alpha^2 - 4\omega^2)$, etc.

4. Equations of state from H.2.

Do the van der Waal problem keeping track of the departures from the ideal gas. The first thing is to get everything into the right variables. Start with

$$\left(P + P_0\frac{\rho^2}{\rho_0^2}\right)\left(V - Na^3\right) = Nk_BT, \quad E = 3k_BT/2m. \tag{22}$$

Re-write

$$\begin{aligned}
\left(P + P_0\frac{\rho^2}{\rho_0^2}\right)\left(1 - \frac{Na^3}{V}\right) &= \frac{N}{V}k_BT, &(23)\\
\left(P + P_0\frac{\rho^2}{\rho_0^2}\right)(1 - \chi(\mu+1)) &= \frac{N}{V}k_BT, &(24)\\
\left(P + P_0\frac{\rho^2}{\rho_0^2}\right) &= \frac{N}{V}k_BT\frac{1}{(1 - \chi(\mu+1))}, &(25)\\
P &= \frac{N}{V}k_BT\frac{1}{(1 - \chi(\mu+1))} - P_0\frac{\rho^2}{\rho_0^2}, &(26)\\
P &= \frac{N}{V}k_BT\,(1 + \chi(\mu+1)) - P_0\frac{\rho^2}{\rho_0^2}, &(27)\\
P_{EOS} &= \rho_0(\gamma-1)(\mu+1)\,(1 + \chi(\mu+1))\,\epsilon - P_0(\mu+1)^2, &(28)\\
P_{EOS} &= \overline{\rho}(\gamma-1)(\mu+1)\epsilon - P_0(\mu+1)^2, &(29)
\end{aligned}$$

where $\chi = \rho_0 a^3/m \ll 1$ and $\overline{\rho} = \rho_0(1 + \chi(\mu + 1))$. Put $\chi = 0$ and $P_0 = 0$ and the ideal gas equation is recovered. Use $\epsilon = \epsilon_0 + u^2/2$ and equate P_{EOS} to $P_H = \rho_0(\mu + 1)u^2/\mu$. Solve for u^2 with the result

$$u^2 = \frac{\frac{\overline{\rho}}{\rho_0}(\gamma - 1)\epsilon_0 - P_0(\mu + 1)}{\frac{1}{\mu} - \frac{\overline{\rho}}{\rho_0}\frac{\gamma - 1}{2}}. \tag{30}$$

A final expression for P results from putting u^2 into P_{EOS} or P_H. Again note that fro $P_0 = 0$ and $\overline{\rho}$ this reduces to the ideal gas result.