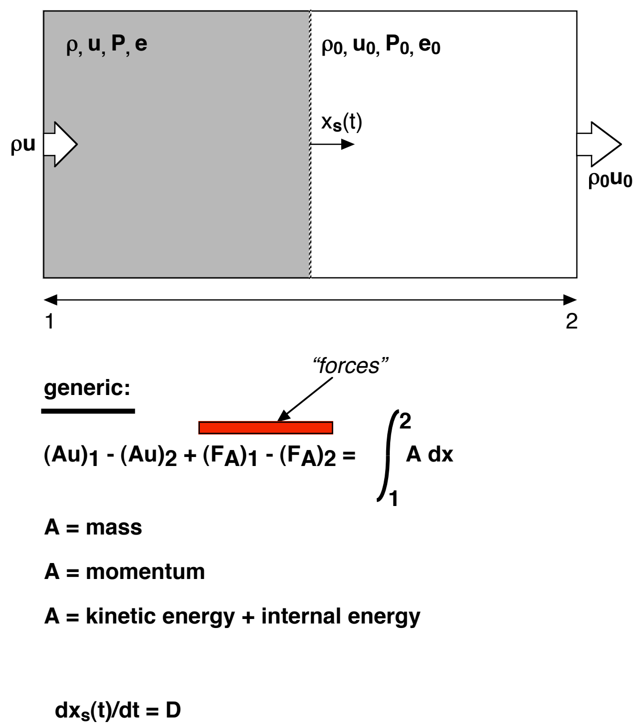

FIG. 1: A shock at $x_s(t)$ as seen in the lab frame. Surfaces 1 and 2 are fixed in the lab frame.

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Shocks. In study of the traffic problem we found a shock feature, defined as a discontinuity in the traffic density (see the I-680 data), that moved with a speed related to the density environment in which it was embedded. In Fig. 1 we show a generalization from this particular example. Consider a piece of material characterized by (a) thermodynamic variables ρ, P and e and (b) the velocity u with which the material is moving in the lab frame. We use subscript 0 for the condition of the material to the right of a shock feature that is moving through the material from left to right. We can form 3 equations that relate the properties of the material to the left of the shock feature to the properties of the material to the right of the shock feature by considering the conservation laws for mass, momentum and energy, the usual suspects. These equations necessarily involve something about the shock feature.

1. Hugoniot Relations.

1. Mass. Mass flows in from the left at rate $(\rho u)_1$ and flows out from the right at rate $(\rho u)_2$. [A factor of the cross sectional area, common to all flow rates, is dropped from each term.] The difference between these two flow rates is the rate at which the mass between 1 and 2 changes with time. We write

$$(\rho u)_1 - (\rho u)_2 = \frac{d}{dt}\int_1^2 dx\ \rho(x). \tag{1}$$

   We break the integral into two parts 1 to $x_s(t)^-$ and $x_s(t)^+$ to 2. The time dependence of the integral arises from the motion of the limit $x_s(t)$ so we have

$$\frac{d}{dt}\int_1^2 dx\ \rho(x) = \rho(x_s(t)^-)D - \rho(x_s(t)^+)D = \rho D - \rho_0 D, \tag{2}$$

   where $D = \dot{x}_s(t)$. As points 1 and 2 are adjacent to the shock feature we have

$$\rho(D-u) = \rho_0(D-u_0). \tag{3}$$

   This is the first of the Hugoniot relations (sometimes Rankine-Hugoniot relations).

2. Momentum. The derivation of the momentum conservation relation is the same as that for the mass with the added feature that momentum can be created by net forces, here the pressures. We write

$$(\rho uu)_1 - (\rho uu)_2 + P_1 - P_2 = \frac{d}{dt}\int_1^2 dx\ \rho(x)u(x). \tag{4}$$

   Handling the integral in the same way as for the mass we have

$$\rho u^2 + P - \rho_0 u_0^2 - P_0 = \rho u D - \rho_0 u_0 D. \tag{5}$$

   This can be re-arranged to read

$$P - P_0 = \rho_0(D-u_0)(u-u_0). \tag{6}$$

   We will call this re-arrangement the second Hugoniot relation.

3. Energy. The derivation of the energy conservation relation is the same as above except (a) the quantity being carried in/out is the kinetic energy plus the internal energy and the "force" is the rate at which the pressure is doing work, creating energy. We write

$$[(\frac{1}{2}\rho u^2+e)u]_1-[(\frac{1}{2}\rho u^2+e)u]_2+(Pu)_1-(Pu)_2=\frac{d}{dt}\int_1^2\ dx\ [\frac{1}{2}\rho(x)u(x)^2+e(x)]. \quad (7)$$

Handling the integral in the same way as above we have

$$(\frac{1}{2}\rho u^2+e+P)u-(\frac{1}{2}\rho_0 u_0^2+e_0+P_0)u_0]=(\frac{1}{2}\rho u^2+e)D-(\frac{1}{2}\rho_0 u_0^2+e_0)D. \quad (8)$$

We will undertake a judicious re-arrangement of this equation below. For the moment we call this equation the third Hugoniot relation.

The point of view we will take about the Hugoniot relations is that they make it possible for us to learn the thermodynamic state of the material on the left of the shock feature. For this to be the case we assume that we know the thermodynamic state and the velocity of the material on the right of the shock feature, i.e., ρ_0, P_0, e_0 and u_0. Looking among the 3 Hugoniot relations, Eqs. (3), (6) and (8), we see that there are 5 unknown variables, ρ, P, e, u and D. Two must be measured. Let's agree to measure D and u, the velocity of the shock and the velocity of the material (called the particle velocity) and have ρ, P and e be the dependent variables.

2. Some useful Algebra and $E-E_0$.

Let's make some re-arrangements of the basic equations that exposes some of their basic content.

1. We can re-arrange Eq. (3) to find the particle velocity difference

$$u-u_0=(D-u_0)\left(1-\frac{\rho_0}{\rho}\right)=\rho_0(D-u_0)(V_0-V), \quad (9)$$

where $V=1/\rho$ is the specific volume, i.e., the volume per unit mass.

2. Use Eq. (9) for $u-u_0$ in Eq. (6) to write

$$\frac{P-P_0}{V_0-V}=\rho_0^2(D-u_0)^2 \quad (10)$$

and Eq. (9) a second time ($\rho_0(D-u_0)=\cdots$) to express $u-u_0$ in terms of P and V

$$(u-u_0)^2=(P-P_0)(V_0-V) \quad (11)$$

or

$$u=u_0+\sqrt{(P-P_0)(V_0-V)}. \quad (12)$$

3. Finally use Eq. (10) to find D

$$(D-u_0)^2 = \frac{1}{\rho_0}\frac{P-P_0}{V_0-V} = V_0^2\frac{P-P_0}{V_0-V} \tag{13}$$

or

$$D = u_0 + V_0\sqrt{\frac{P-P_0}{V_0-V}}. \tag{14}$$

4. In the energy equation, Eq. (8) use the specific energy defined by $e = \rho E$ to write

$$\rho_0(D-u_0)\left(E - E_0 + \frac{u^2}{2} - \frac{u_0^2}{2}\right) = Pu - Pu_0 \tag{15}$$

and re-arrange to find E

$$E - E_0 = \frac{Pu - Pu_0}{\rho_0(D-u_0)} - \left(\frac{u^2}{2} - \frac{u_0^2}{2}\right) \tag{16}$$

   (a) for $\rho_0(D-u_0)$ use Eq. (10),

   (b) use u from Eq. (12) to express $Pu - P_0u_0$ in the form

$$Pu - P_0u_0 = u_0(P-P_0) + P\sqrt{(P-P_0))V_0 - V)}, \tag{17}$$

   (c) use u from Eq. (12) to express the kinetic energy difference in the form

$$\left(\frac{u^2}{2} - \frac{u_0^2}{2}\right) = \frac{1}{2}(P-P_0)(V_0-V) + u_0\sqrt{(P-P_0)(V_0-V)}. \tag{18}$$

When these pieces are assembled the result is

$$E - E_0 = \frac{P+P_0}{2}\left(V_0 - V\right). \tag{19}$$

The three quantities in this equation are the derived variables that describe the thermodynamic state of the material to the left of the shock feature. What this equations says is that the Hugoniot relations imply a sensible connection between the change in internal energy and some average work done on the material by the pressures adjacent to the shock.

3. $P-V$ Space.

For convenience choose $u_0 = 0$. Consider $P-V$ space shown in Fig. 2. The two points (V_0, P_0) and (P, V) on opposite sides of the shock feature can be placed in this space. Join the points by the line

$$P = P_0 + \left(\frac{dP}{dV}\right)_D (V - V_0) = P_0 + D\rho_0^2(D-u_0)^2(V_0 - V) \tag{20}$$

from Eq. (10). This line is called the Rayleigh line.

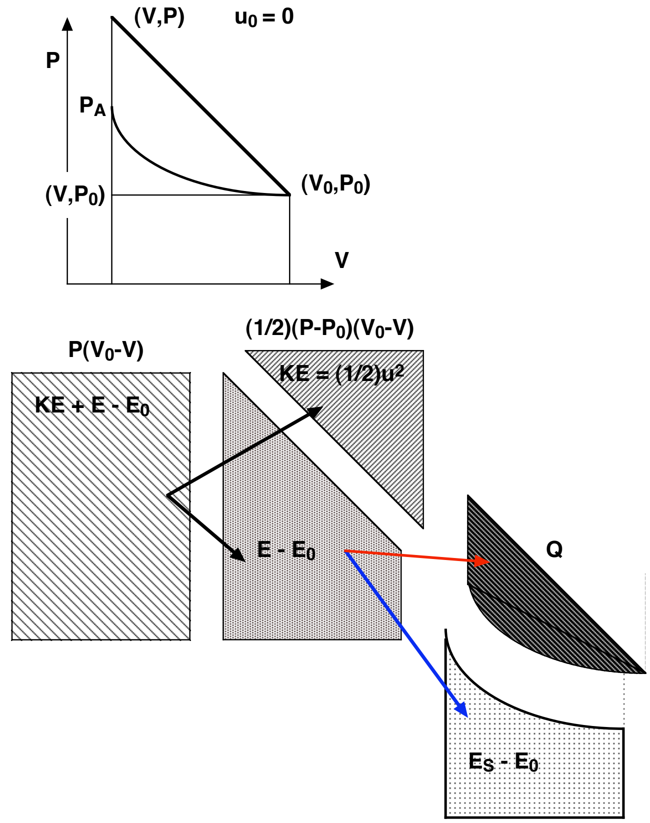

FIG. 2: P-V space and various pieces of it.

1. From Eq. (11), with $u_0 = 0$ we have

$$\frac{u^2}{2} = \frac{1}{2}(P - P_0)(V_0 - V), \tag{21}$$

i.e., the kinetic energy is the area within the triangle $(V_0, P_0) \to (P, V) \to (P_0, V) \to (V_0, P_0)$.

2. Combine Eq. (21) with Eq. (19) and find

$$\frac{u^2}{2} + E - E_0 = P(V_0 - V). \tag{22}$$

The area of the rectangle (crosshatch in Fig. 2) is the energy difference between the material to the left and right of the shock feature.

3. The difference in internal energy between the material to the left and right of the shock feature = rectangle − triangle.

4. Suppose compression from V_0 to V were undertaken adiabatically (by some means). The final pressure point P_A and trajectory in $P - V$ space might be that shown by the curved line from P_0 to P_A. The change in internal energy of the material on the left of the shock feature is $E_S - E_0$ associated with the area shown in the lower right of Fig. 2.

5. The difference between E and E_S is due to energy flow, Q, into the system as the shock feature propagates. Thus the change in the energy of the material due to heating is represented by the dark area to the right in Fig. 2.

These interpretations are quantitative when the quantities necessay to calculate the areas in the $P - V$ space are available. For the most part the variables used in this discussion have been the dependent variables in the description. Some of these are found from D and u and the properties of the material to the right of the shock feature. Other are found by modeling the material. Regardless of the quantitative use of these $P - V$ area identifications, their qualitative behavior can aid in understanding.