Physics 740: Spring 2006: 01/22/07

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I. An example. [This is out of the clear blue sky. Expect to understand this in detail later.]

I.A There are two equations that describe a fluid, the continuity equation and the Navier -Stokes equation. The first is a statement of the conservation of matter and the second is a statement of the conservation of momentum (the analogue of Newton II). The variables in these equations are density, ρ, velocity, $\mathbf{v}$, pressure, P and such. The two equations are

1. continuiiy
$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0, \tag{1}$$

2. Navier-Stokes (approximately)
$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} = -\frac{1}{\rho} \nabla P + \eta \nabla^2 \mathbf{v}. \tag{2}$$

Remarks and observations:

1. We are not using microscopic variables, particle mass, particle velocity, etc. The variables involved are **coarse grained**; ρ, P, $\mathbf{v}$, $\cdots$ are associated with a differential element of volume, $dV = dxdydz$, in which the mass is $dm = \rho dV$, the velocity $\mathbf{v}$ is the average of the velocity of the particles in the volume, etc.

2. These equations are **nonlinear**. What does this mean?

3. Count variables and equations. If we agree that η is a viscosity with known value only ρ, P and $\mathbf{v}$ are left. So in D dimensions that's $N_V = D + 2$ variables and $N_E = D + 1$ equations. We might close these equations ($N_V = N_E$) by asserting that we know P as a function of ρ, $P = \rho k_B T/m$ or some such. Rule of this type are *equations of state*, the province of **thermodynamics**! Yuk!!! You may have to know some thermo to get very far with the N-S equation?

4. If you check the dimensions of η they are L^2/T, like a diffusion constant. The viscosity is a *transport coefficient*. Transport coefficients describe the transport of something. In the case of a diffusion coefficient it would be a particle current, e.g., $\mathbf{J} = -D\nabla n$. You can formulate a description of transport processes in non-equilibrium **statistical**

physics. *Non-equilibrium* because you have to continuously do work to maintain the (particle) current. For a more familiar example, think about an electric current.

From these observations a rough outline for the semester is suggested, Stat. Mech. (4 week), Thermo. (4 weeks), Transport Phenomena (4 weeks), that leaves 2 weeks to do Fluid Mechanics. As the text is 531 pages in length the pace towrd the end of the course will be quite rapid. More about the outline below. But first let's get something out of Eqs. (1) and (2).

I.B Suppose Eqs. (1) and (2) described a fluid that in near equilibrium at $(\rho_0, P_0, \mathbf{v}_0 = 0)$ throughout space. Small local departures from equilibrium would involve $\rho = \rho_0 + \delta\rho$, $P = P_0 + \delta P$ and $\mathbf{v} = \mathbf{v}_0 + \delta\mathbf{v} = \delta\mathbf{v}$, where $\delta\rho(\mathbf{x}, t)$, $\delta P(\mathbf{x}, t)$ and $\delta\mathbf{v}(\mathbf{x}, t)$ are first order. If we decide to linearize Eqs. (1) and (2) we would drop the $(\delta\mathbf{v} \cdot \nabla\delta)\mathbf{v}$ as second order, replace $\rho^{-1}\nabla P$ by $\rho_0^{-1}\nabla\delta P$, etc. with the results

$$\frac{\partial\delta\rho}{\partial t} + \rho_0\nabla \cdot (\delta\mathbf{v}) = 0, \tag{3}$$

and

$$\frac{\partial\delta\mathbf{v}}{\partial t} = -\frac{1}{\rho_0}\nabla\delta P + \eta\nabla^2\delta\mathbf{v}. \tag{4}$$

We still have $N_V \neq N_E$. Your local thermodynamicist will provide you with $P(\rho)$. Use this relation near $(\rho_0, P_0, \mathbf{v}_0 = 0)$, $\delta P = (\partial P/\partial\rho)_0\ \delta\rho$. If you check the dimensions of $(\partial P/\partial\rho)_0$ you should find it has those of a velocity squared. So define $c_0^2 = (\partial P/\partial\rho)_0$. Then the pair of equations is

$$\frac{\partial\delta\rho}{\partial t} + \rho_0\nabla \cdot (\delta\mathbf{v}) \ = \ 0, \tag{5}$$

$$\frac{\partial\delta\mathbf{v}}{\partial t} + \frac{c_0^2}{\rho_0}\nabla\delta\rho - \eta\nabla^2\delta\mathbf{v} \ = \ 0. \tag{6}$$

The quantities ρ_0, c_0^2 and η are presumed known and are parameters in what we are doing. The equations are linear in $\delta\rho$ and $\delta\mathbf{v}$ which are to be found. The equations are homogeneous, i.e., there is nothing that forces either $\delta\rho$ or $\delta\mathbf{v}$ to be non-zero, like $m\ddot{x} = -Kx$ but not like $m\ddot{x} = -Kx + F\sin\omega_0 t$. While $\delta\rho = \delta\mathbf{v} = 0$ is a solution to Eqs. (5) and (6) a more interesting discussion follows if one assumes $\delta\rho$ and $\delta\mathbf{v}$ are non-zero and wonders how they are related to one another. Guess

$$\delta\rho \ = \ Ae^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}, \tag{7}$$

$$\delta\mathbf{v} \ = \ \mathbf{B}e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}, \tag{8}$$

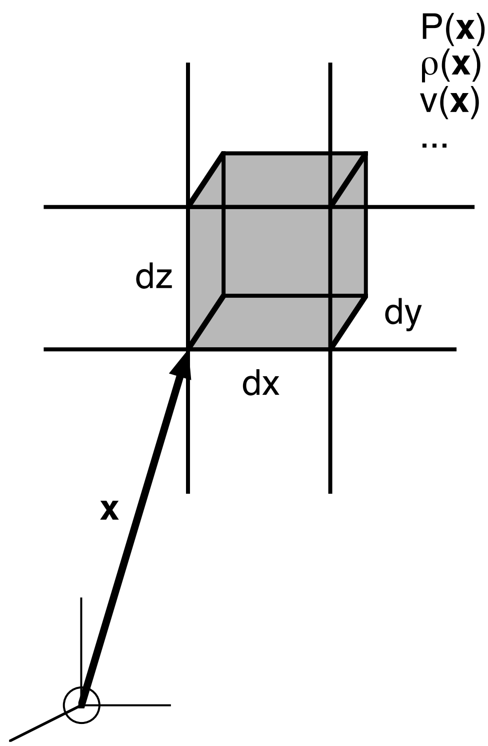

FIG. 1: Coarse Grained. Fluid dynamic variables are local values of things like ρ, P, $\mathbf{v}$, $\cdots$ that are assigned to differential elements of volume, $dV = dxdydz$. If dV is too large it is not useful, e.g., the right hand side of the Atlantic ocean. If dV is too small it is not useful, e.g., a cubic Angstrom in a room temperature, atmospheric pressure gas. A dV of $(1\ mm)^3$ has a fluctuation in $n = \rho/m$ of about 1 part in 10^8 (room temperature, atmospheric pressure). Why?

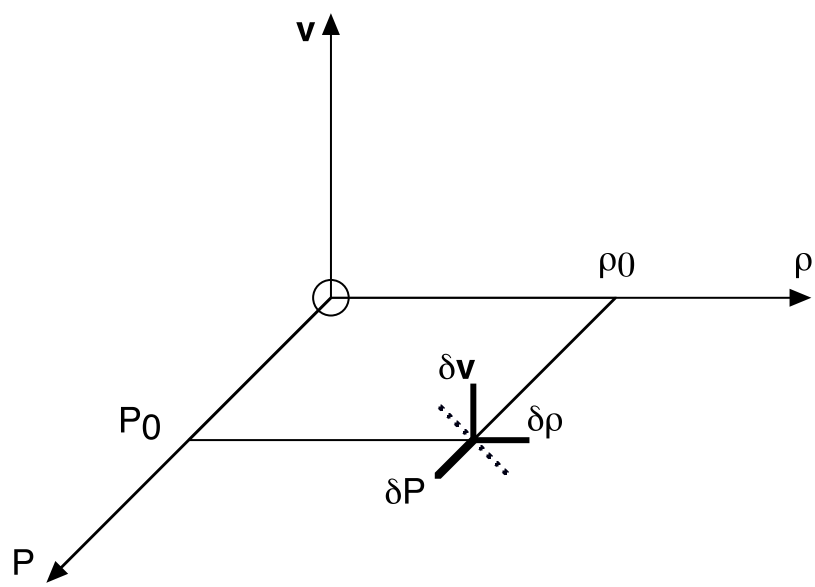

FIG. 2: Fluctuation Away From Equilibrium. At $(\rho_0, P_0, \mathbf{v}_0 = 0)$ there are small departures in ρ, P and $\mathbf{v}$. These *fluctuatons* may be spatially local, depend on $\mathbf{x}$, and can change in time. In equilibrium $\delta\rho = \delta P = \delta\mathbf{v} = 0$.

and substitute into Eqs. (5) and (6). [If this is a bad idea you won't find anything interesting and you'll have to try something else.] So do the algebra and find

$$-i\omega A + \rho_0\, i\mathbf{k}\cdot\mathbf{B} \;=\; 0, \tag{9}$$

$$-i\omega\mathbf{B} + \frac{c_0^2}{\rho_0} i\mathbf{k}A + \eta k^2\mathbf{B} \;=\; 0. \tag{10}$$

Define $C = i\mathbf{k}\cdot\mathbf{B}$ and form this quantity in the second equation

$$-i\omega A + \rho_0 C \;=\; 0, \tag{11}$$

$$-i\omega C + \frac{c_0^2}{\rho_0} k^2 A + \eta k^2 C \;=\; 0. \tag{12}$$

Solution to this pair of equations leads to a relationship between the frequency ω and the wavevector $\mathbf{k}$

$$\omega^2 = c_0^2 k^2 - i\eta\omega k^2 \tag{13}$$

and a relationship between the size of the two fluctuations,

$$A = \rho_0 \frac{\mathbf{k}\cdot\mathbf{B}}{\omega}. \tag{14}$$

This last equation is essentially $\delta\rho/\rho_0 \approx |\delta\mathbf{v}|/c_0$.

II Syllabus: very approximate.

Part I; Introduction

1. Review of useful stuff (1 week)
   (a) Stat. Mech.
   (b) Thermo
   (c) Getting transport coefficients
2. Liousville equation to Boltzmann equation (2 weeks)
   (a) "Derive" Boltzmann equation
   (b) Get Euler and Navier-Stokes from Boltzmann
   (c) Numerical implementation of Boltzmann
3. Navier-Stokes, etc. from conservation laws (1 week)

4. A look at computational fluid dynamics (1 week)

Part II: Topics of Interest

Various items from the first 9 chapters of L and L (possibly including) shallow/deep water waves, tsunami, KdV solitons, shock waves, onset of turbulence, a curve ball, ”slice”?, traffic, oscillons, and topics that arise.