HW5, Fluid Dynamics

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0.1 Problem #2

A shallow channel has V-shaped cross section with breadth $2a$ and depth h_0 at the center , $x = 0$.

0.1.1 Modes across the channel

Show that the modes that flow back and forth across the channel (they are uniform along the channel) have even or odd symmetry about $x = 0$.

$\delta h(x,t)$ denotes surface fluctuations from h_0. To get a wave equation, we have to write down 2 equations - fluctuations in time and in space. Let's begin with time dependence and consider a region of fluid, which is really small in x direction. Equation of continuity states that the amount of fluid flowing into the region must be equal to the amount of fluid flowing out of the region. So for V shaped channel we can write the equation:

$$\begin{aligned} \frac{\partial \delta h}{\partial t} + h_0 \frac{\partial}{\partial x}\left(\left(1 - \frac{|x|}{a}\right) v(x)\right) &= 0, \\ \frac{\partial \delta h}{\partial t} + h_0 \frac{v(x)}{a} + h_0 \left(1 + \frac{x}{a}\right) \frac{\partial v(x)}{\partial x} &= 0. \end{aligned} \tag{1}$$

Notice that from here further on we consider only the region $x < 0$. The next equation could be found from Euler equation for incompressible fluids:

$$\frac{\partial v}{\partial t} = -\frac{1}{\rho_0} \nabla P.$$

We are interested in wave movement in x direction only, because $\frac{\partial v_z}{\partial t} \approx 0$. So for z component of the gradient we can write

$$\frac{\partial P}{\partial z} = \rho_0 g \rightarrow P = P_0 + \rho_0 g (\delta h(x) - z).$$

For the v_x the equation

$$\frac{\partial v_x}{\partial t} = -\frac{1}{\rho_0} \rho_0 g \frac{\partial \delta h(x)}{\partial x} \Rightarrow \frac{\partial v_x}{\partial t} + g \frac{\partial \delta h(x)}{\partial x} = 0 \tag{2}$$

must hold. So, by multiplying Eq. (1) by $\frac{\partial}{\partial t}$ and Eq. (2) by $\frac{\partial}{\partial x}$, we are able to obtain a wave equation for δh:

$$\frac{\partial^2 \delta h}{\partial t^2} + h_0 \left[\frac{1}{a} \frac{\partial v(x)}{\partial t} + \left(1 + \frac{x}{a}\right) \frac{\partial^2 v(x)}{\partial t \partial x}\right] = 0, \tag{3}$$

$$\frac{\partial v_x^2}{\partial x \partial t} + g \frac{\partial^2 \delta h(x)}{\partial x^2} = 0. \tag{4}$$

By substituting first term of Eq. (4) and Eq. (2) into the Eq. (3), the wave equation gets the form:

$$\frac{\partial^2 \delta h}{\partial t^2} - g h_0 \left[\frac{1}{a} \frac{\partial \delta h}{\partial x} + \left(1 + \frac{x}{a}\right) \frac{\partial^2 \delta h}{\partial x^2}\right] = 0. \tag{5}$$

To solve the equation, we make a substituion $z = 1 + \frac{x}{a}$. So $\frac{\partial z}{\partial x} = \frac{1}{a}$:

$$\frac{\partial^2 \delta h}{\partial t^2} - gh_0 \left[\frac{1}{a^2} \frac{\partial \delta h}{\partial z} + z \frac{\partial^2 \delta h}{\partial x^2} \right] = 0$$

For steady state solution, $\delta h = H(z)cos(\omega t)$. By substituting this into the equation:

$$-\omega^2 H - \frac{gh_0}{a^2} \left(\frac{\partial H}{\partial z} + z \frac{\partial^2 H}{\partial z^2} \right) = 0 | \cdot \left(-\frac{za^2}{c_0^2} \right),$$

where $c_0^2 = gh_0$. The manipulation results in

$$z^2 \frac{\partial^2 H}{\partial z^2} + z \frac{\partial H}{\partial z} + \frac{\omega^2 a^2}{c_0^2} zH = 0,$$

where also $k^2 = \omega^2 a^2 / c_0^2$ holds true. Basically this is a bessel equation in a form of:

$$x^2 \frac{d^2 H}{dx^2} + (2p + 1) x \frac{dH}{dx} + \left(\alpha^2 x^{2r} + \beta^2 \right) H = 0,$$

with a solution

$$H = x^{-p} \left[C_1 J_{q/r} \left(\frac{\alpha}{r} x^r \right) + C_2 Y_{q/r} \left(\frac{\alpha}{r} x^r \right) \right],$$

where $q \equiv \sqrt{p^2 - \beta^2}$. For our case, the solution would be:

$$H = C_1 J_0 \left(2k\sqrt{z} \right) = C_1 J_0 \left(\frac{2\omega a}{c_0} \sqrt{\left(1 + \frac{x}{a} \right)} \right). \tag{6}$$

As we are looking for even and odd solution at the point $x = 0$, we have to look for solution $J_0 = 0$ or $\frac{dJ_0}{dx} = 0$ at the point $x = 0$. So for odd function we have to find ω values for $J_0 = 0$ (denoted as arg_0) and for even function ω values for $\frac{dJ_0}{dx} = 0$ (denoted as $argd_0$). For that we have to solve correspondingly equations $arg_0^{(n)} = \frac{2\omega^{(n)} a}{c_0}$ and $argd_0^{(n)} = \frac{2\omega^{(n)} a}{c_0}$, where $n = 1...6$. The corresponding values for Bessel function and ω values are in Table 1.

0.2 Problem #2

Bounded oil/water. Find the dispersion relation for the waves on the interface of 2 fluids. Fluid 1, density ρ_1, resides $0 \leq z \leq h_1$ above fluid 2, density $\rho_2 > \rho_1$, that resides in $-h_1 \leq z \leq 0$. Gravity is at work.

zeros of J_0	zeros of J_0'	odd ω values	even ω values
$arg_0^{(1)} = 2.404$	$argd_0^{(1)} = 3.832$	$\omega^{(1)} = 1.202\frac{c_0}{a}$	$\omega_d^{(1)} = 1.916\frac{c_0}{a}$
$arg_0^{(2)} = 5.520$	$argd_0^{(2)} = 7.016$	$\omega^{(2)} = 2.76\frac{c_0}{a}$	$\omega_d^{(2)} = 3.508\frac{c_0}{a}$
$arg_0^{(3)} = 8.654$	$argd_0^{(3)} = 10.174$	$\omega^{(3)} = 4.327\frac{c_0}{a}$	$\omega_d^{(3)} = 5.087\frac{c_0}{a}$
$arg_0^{(4)} = 11.792$	$argd_0^{(4)} = 13.324$	$\omega^{(4)} = 5.896\frac{c_0}{a}$	$\omega_d^{(4)} = 6.662\frac{c_0}{a}$
$arg_0^{(5)} = 14.931$	$argd_0^{(5)} = 16.471$	$\omega^{(5)} = 7.466\frac{c_0}{a}$	$\omega_d^{(5)} = 8.236\frac{c_0}{a}$
$arg_0^{(6)} = 18.071$		$\omega^{(6)} = 9.036\frac{c_0}{a}$	

Table 1: Zeros of Bessel function and its derivative. Notice that we could have hidden the term $\frac{c_0}{a}$ into the variable k, but now it comes out really nicely how the ω is related to maximu depth and width of the water channel.

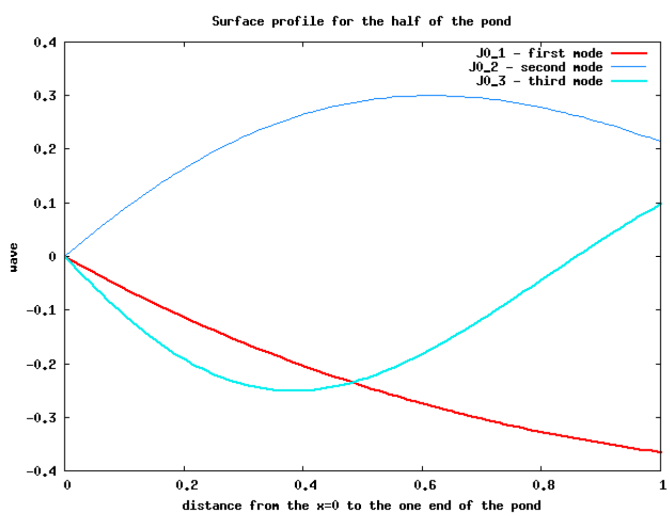

Figure 1: 3 modes of the surface profile for the case that $J_0 = 0$ at $x = 0$.

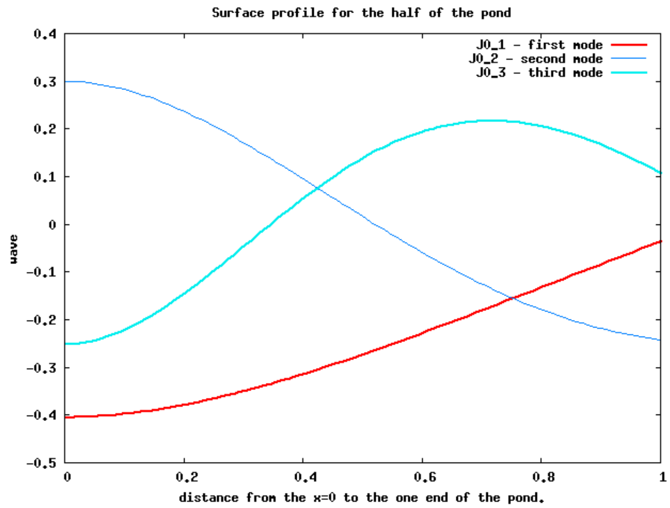

Figure 2: 3 modes of the surface profile for the case that $\frac{dJ_0}{dx} = 0$ at $x = 0$.

For potential flow we can write for the surface of separation

$$\rho_1 g \delta h + \rho_1 \frac{\partial \phi_1}{\partial t} = \rho_2 g \delta h + \rho_2 \frac{\partial \phi_2}{\partial t} \tag{7}$$

For ϕ we can look for a solution as was found in the class (for waves in general):

$$\phi_1 = A \cdot cosh\left(kz - kh_1\right) cos\left(kx - \omega t\right) \tag{8}$$
$$\phi_2 = B \cdot cosh\left(kz + kh_2\right) cos\left(kx - \omega t\right) \tag{9}$$

From the Eq. 7 we can find δh:

$$\delta h = \frac{1}{g\left(\rho_1 - \rho_2\right)} \left(\rho_2 \frac{\partial \phi_2}{\partial t} - \rho_1 \frac{\partial \phi_1}{\partial t}\right).$$

Also as liquid is continuos, the following equations must hold true in the separation layer:

$$\frac{\partial \phi_1}{\partial z} = \frac{\partial \phi_2}{\partial z} \tag{10}$$
$$v_z = \frac{\partial \phi_1}{\partial z} = \frac{\partial \delta h}{\partial t}$$

So the previous equation comes by multiplying by $\frac{\partial}{\partial t}$:

$$\frac{\partial \phi_1}{\partial z} g\left(\rho_1 - \rho_2\right) = \rho_2 \frac{\partial^2 \phi_2}{\partial t^2} - \rho_1 \frac{\partial^2 \phi_1}{\partial t^2} \tag{11}$$

Lets substitute Equations (8) and (9) into the obtained quations (10) and (11):

$$\begin{aligned} A\cdot k\cdot sinh\left(kz-kh_1\right)g\left(\rho_1-\rho_2\right) &= -\omega^2\rho_2 B\cdot cosh\left(kz+kh_2\right)+\omega^2\rho_1 A\cdot cosh\left(kz-kh_1\right) \\ -A\cdot sinh\left(kh_1\right) &= B\cdot sinh\left(kh_2\right) \end{aligned}$$

The latter equations is true for the case $z=0$, so we'll get:

$$\begin{aligned} B &= -\frac{A\cdot sinh\left(kh_1\right)}{sinh\left(kh_2\right)} \\ -k\cdot sinh\left(kh_1\right)g\left(\rho_1-\rho_2\right) &= \omega^2\rho_2\frac{sinh\left(kh_1\right)}{sinh\left(kh_2\right)}\cdot cosh\left(kh_2\right)+\omega^2\rho_1 cosh\left(kh_1\right) \\ \omega^2 &= \frac{-k\cdot sinh\left(kh_1\right)g\left(\rho_1-\rho_2\right)}{\rho_1 cosh\left(kh_1\right)+\rho_2\cdot sinh\left(kh_1\right)\cdot coth\left(kh_2\right)} \\ \omega^2 &= \frac{k\cdot g\left(\rho_2-\rho_1\right)}{\rho_1 coth\left(kh_1\right)+\rho_2\cdot coth\left(kh_2\right)}. \end{aligned}$$

So we have found dispersion relation for layer of separation of two fluids.

0.2.1 Various limits

- $\rho_2\to\rho_1$ Easy to see that dispersion relations relation approaches to 0. Could be explained by the fact that there is now wave propagating in the middle of fluid for that case.
- $\rho_2\gg\rho_1$For example if ρ_1 is air. We'll get the well known relation for the fluid with free surface: $\omega^2=k\cdot g\cdot tanh\left(kh_2\right)$. Also for really long wavelength ($kh_2\ll 1$) the relation will be $\omega^2=gh_2k^2$as found in class also. Similarly for the opposite case the $\omega^2=gk$.
- $\rho_2\ll\rho_1$Actually we assumed that $\rho_2>\rho_1$. But if it is not the case, then the $\omega^2<0$. It could mean that this configuration is not stable for wave propagation.

0.3 Problem #3

Find the Fourier transform of

$$S\left(t;t_0,\Delta t,\Omega\right)=S(0)\frac{1}{\sqrt{2\pi\left(\Delta t\right)^2}}e^{-\frac{1}{2}\frac{\left(t-t_0\right)^2}{\left(\Delta t\right)^2}}sin\Omega\left(t-t_0\right)$$

It is good to consider following useful relations for finding the transform:

- $g\left(t-a\right)\Rightarrow e^{-ia\omega}G\left(\omega\right)$
- $f\left(t\right)sin\left(\Omega t\right)\Rightarrow\frac{i}{2}\left(F\left(\omega+\Omega\right)-F\left(\omega-\Omega\right)\right)$
- $e^{-\alpha t^2}\Rightarrow\frac{1}{\sqrt{2\alpha}}e^{-\frac{\omega^2}{4\alpha}}$

So the transform of the gaussian part of $S(t)$ (shifted to 0) would be:

$$\boldsymbol{F}\left(e^{-\frac{1}{2}\frac{t^2}{(\Delta t)^2}}\right)=\sqrt{(\Delta t)^2}e^{-\frac{\omega^2}{2}(\Delta t)^2}$$

Now lets consider the sine term and constant multipliers also:

$$S_0(\omega)=S(0)\frac{i}{2\sqrt{2\pi}}\left(e^{-\frac{(\omega+\Omega)^2}{2}(\Delta t)^2}-e^{-\frac{(\omega-\Omega)^2}{2}(\Delta t)^2}\right)$$

Now let's shift the result also by t_0:

$$S(\omega)=S(0)\frac{i\cdot e^{-it_0\omega}}{2\sqrt{2\pi}}\left(e^{-\frac{(\omega+\Omega)^2}{2}(\Delta t)^2}-e^{-\frac{(\omega-\Omega)^2}{2}(\Delta t)^2}\right),$$

which is the final form of the transform.

0.4 Problem #4

At $t=0$, the displacement of a shallow water wave is

$$\delta h_0(x)=Cexp\left(-x^2/\left(2W^2\right)\right),$$

where $W=0.5$ and C is fixed by $\int\delta h(x)dx=1$. Find the location and shape of this disturbance at later times. Assume that the dispersion relation for water waves is (the almost shallow case)

$$\omega=c_0k\left(1-\alpha k^2\right),$$

where $\alpha=0.0001$. The answer is basically

$$\delta h(x,t)=\int\frac{dk}{2\pi}\delta h(k)exp(i(kx-\omega(k)t)$$

where

$$\delta h(k)=\int dx\delta h(x)exp(-ikx)$$

Basically the solution could be found taking inverse Fourier transform of the equation

$$\delta h(x,t)=\int\frac{dk}{2\pi}\delta(k,t)exp(ikx)$$

where

$$\delta h(k,t)=\delta h(k)exp(-i\omega(k)t).$$

The tricky part is that after finding the FFT of for $\delta h(k)$, we have to consider how the components are placed in the array in MATLAB. So we cannot just multiply by $exp(-i\omega(k)t)$. For first half (and +1) of the array we can multiply really as usually: component wise. But for second half of the array we have to consider that there are complex conjugates in $\delta h(k)$ array and ordered vice versa. So the second half of the array must also be multiplied by complex conjugates of exponent term and also the component arrangement must be considered. The results are commented in Figures:

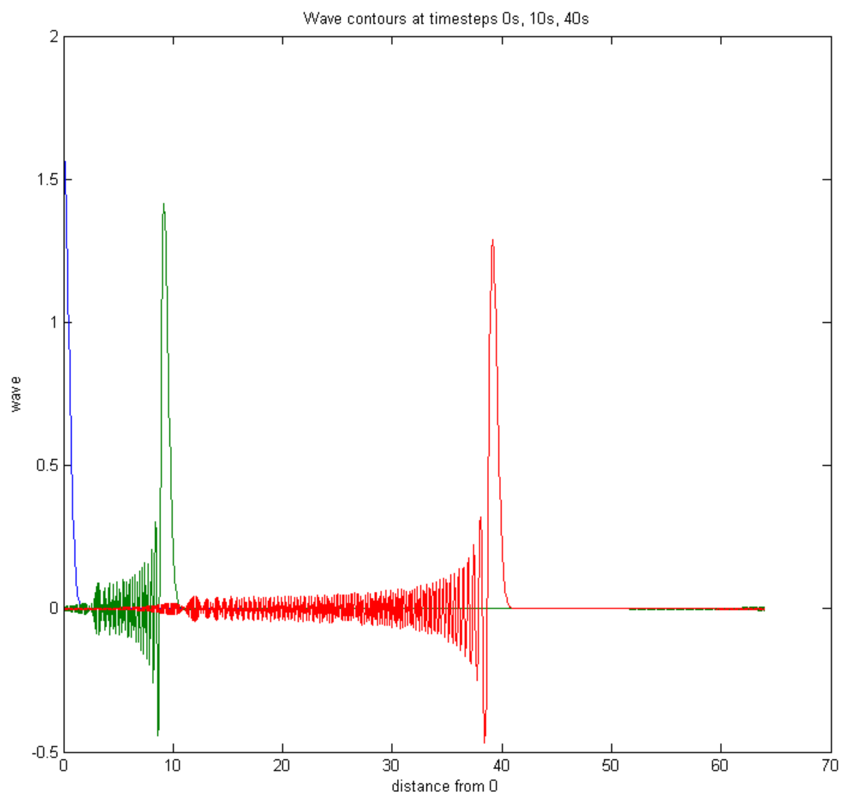

Figure 3: Wave shape at 3 different times. Notice the heavy wake behind the wave. It is caused by the fact that initial wave was HALF gaussian, i.e. gaussian located at the 0, but only positive x was considered. Due to the dispersion the wave amplitude is also decreasing and the wake is increasing.

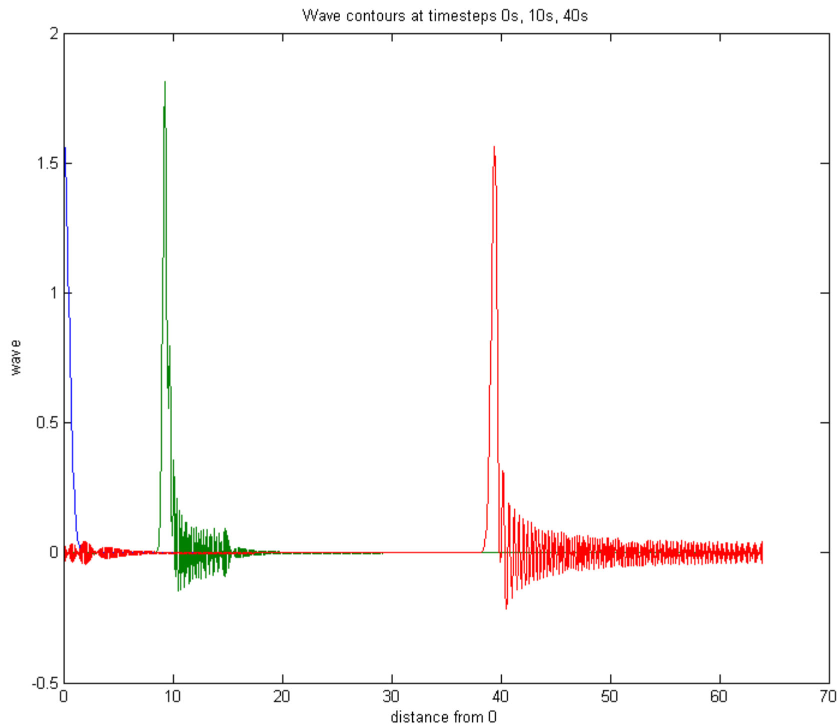

Figure 4: The situation is same as in Figure 3. Only the α is negative this time. Notice that wake is in the front of the wave.

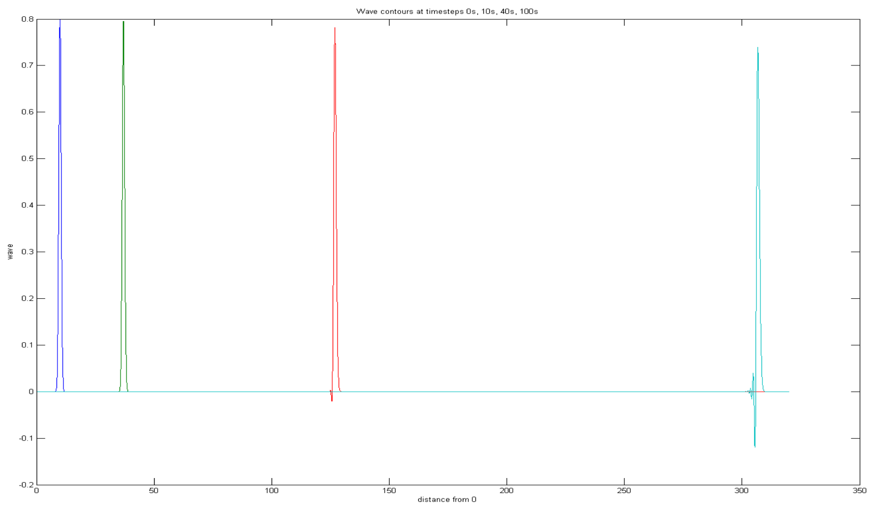

Figure 5: Now we have shifted the gaussian to the right by -5. Notice that there is not heavy wake behind because this kind of gaussian consists less harmonics, so less dispersive also.

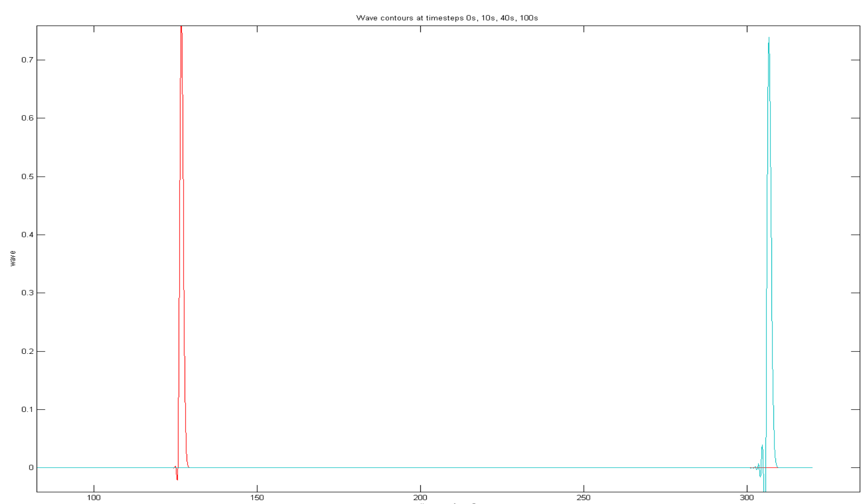

Figure 6: Magnified Figure 5.

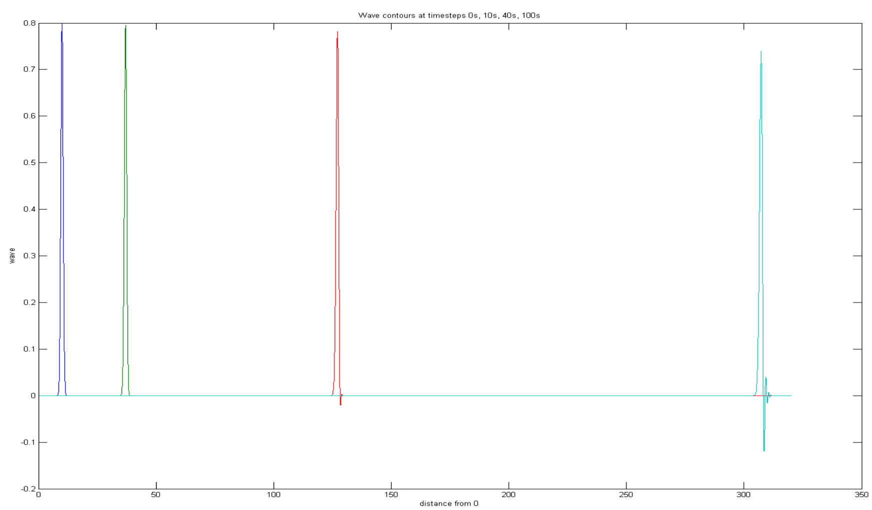

Figure 7: The situation is same as in Figure 5. Only $\alpha < 0$. So now the wake is in the front of the wave.