Physics 740: Spring 2007: 03/28/07

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Waves II: Dispersion and Nonlinearity.

1. Review. We will use *water* waves as an example (*water* because this is really an example of a class of systems characterized by the free surface of the fluid; a well known other example is the *third sound* mode of a superfluid thin film.)

2. Shallow water. The back of the envelope argument gave

$$\frac{\partial \delta h}{\partial t} + h_0 \nabla \cdot \mathbf{v} = 0 \tag{1}$$

and

$$\frac{\partial \mathbf{v}}{\partial t} = -g \nabla \delta h \tag{2}$$

which combine to

$$\frac{\partial^2 \delta h}{\partial t^2} = g h_0 \nabla^2 \delta h \tag{3}$$

from which we get the speed of a tsunami

$$c_0^2 = g h_0. \tag{4}$$

3. Deep water. In the deep water case we gave a more formal treatment of the fluid.

$$\nabla^2 \phi \;=\; 0, \tag{5}$$

$$\mathbf{n} \cdot \mathbf{v} \;=\; 0, \quad z = -h_0, \tag{6}$$

$$\frac{\partial \hat{\phi}}{\partial z} \;=\; -\frac{1}{g}\frac{\partial^2 \hat{\phi}}{\partial t^2}, \quad z = 0. \tag{7}$$

Solution to this system of equations leads to

$$\phi(x,z,t) = A_0\ coshk(h_0 - z)\ cos(kx)\ cos(\omega t) \tag{8}$$

and the dispersion relation

$$\omega^2 = gk \tanh kh_0. \tag{9}$$

From Eq. (9) we have

1. shallow water, $kh_0 \ll 1$,

$$\omega^2 = gh_0 k^2 = c_0^2 k^2. \tag{10}$$

2. almost shallow water, $kh_0 < 1$,

$$\omega^2 = c_0^2 \left(1 - \frac{1}{3}(kh_0)^2 + \cdots\right) k^2 \tag{11}$$

3. deep water, $kh_0 \gg 1$,

$$\omega^2 = gk. \tag{12}$$

We will return to case 2 below, almost shallow.

4. Dispersion relations and Fourier analysis. From Eq. (11) we have two forms of the dispersion relation (for almost shallow water)

$$\omega(k) = c_0 k \left(1 - \frac{1}{6}(kh_0)^2 + \cdots\right) \tag{13}$$

or equivalently

$$k(\omega) = \frac{\omega}{c_0}\left(1 + \frac{1}{6}\left(\frac{\omega h_0}{c_0}\right)^2 + \cdots\right). \tag{14}$$

Note that both ω and k can change sign and that $k(-\omega) = -k(\omega)$ and $\omega(-k) = -\omega(k)$. Dispersion relations of this type are often encountered in the Fourier treatment of a linear system. In general you may have Fourier representation of $f(x,t)$ in the form

$$f(x,t) = \int \frac{dk}{2\pi} \int \frac{d\omega}{2\pi}\, A(k,\omega) e^{i(kx-\omega t)}, \tag{15}$$

where the limits on k and ω are $\pm\infty$ and factors of 2π are here. The inverse transforms have no factors of 2π. When you have a dispersion relation you are able to write either

$$A(k,\omega) = A(\omega) 2\pi\delta(k - k(\omega)) \tag{16}$$

or

$$A(k,\omega) = A(k) 2\pi\delta(\omega - \omega(k)). \tag{17}$$

1. If you are doing signal processing you are likely to prefer featuring ω, Eq. (16).

2. If $f(x,t)$ is a real function then

$$A(-\omega) = A(\omega)^*, \tag{18}$$

$$A(-k) = A(k)^*. \tag{19}$$

3. If the dispersion relation has no dispersion, $\omega = c_0 k$, then

$$f(x,t) = \int \frac{d\omega}{2\pi} A(\omega) e^{-i\omega(t - c_0^{-1} x)} = f(x - c_0 t). \tag{20}$$

5. Practical Fourier Analysis. Fast Fourier Transform = FFT. [James W. Cooley and John W. Tukey, "An algorithm for the machine calculation of complex Fourier series," Math. Comput. **19**, 297 - 301 (1965).] Since the invention of the FFT algorithm Fourier transforms have become a very important tool in numerical analysis. **Fast** is the point. Let us use k as example. The real line $0 < x < +L$ is covered by N points. (Unless it cannot be helped there is no point in having N not be a power of 2.) The set of N space points are

$$x_n = (n-1)\Delta x, \quad \Delta x = L/N, \quad n = 1 \cdots N. \tag{21}$$

Corresponding to these are N values of the wavevector

$$k_m = 2\pi(m-1)\frac{1}{N\Delta x}, \quad m = 1 \cdots N. \tag{22}$$

For the function $f(x)$ defined at the N space points, i.e., $f(x_n) = f_n,\ n = 1 \cdots N$, the value of the Fourier transform at wavevector k_m is

$$F_m = F(k_m) = \sum_{n=1}^{N} f_n \, e^{-i2\pi(m-1)(n-1)/N}. \tag{23}$$

If the Fourier transform of a function defined at the points $x_1 \cdots x_n$ is known, i.e., all of the numbers (typically complex) $F_1 \cdots F_N$ are known, then the function of x corresponding to this Fourier transform is given by the inverse transform

$$f_n = f(x_n) = \frac{1}{N} \sum_{n=1}^{N} F_m \, e^{i2\pi(n-1)(m-1)/N}. \tag{24}$$

Usually, in Mathematica, Matlab, Mathcad, Maple, ... there are intrinsic functions that do the calculations above. You need only supply as input the set of values of $f(x)$, i.e., $f_1 \cdots f_N$, to use Eq. (23) and the set of values of $F(k)$, i.e., $F_1 \cdots F_N$, to use Eq. (24). There

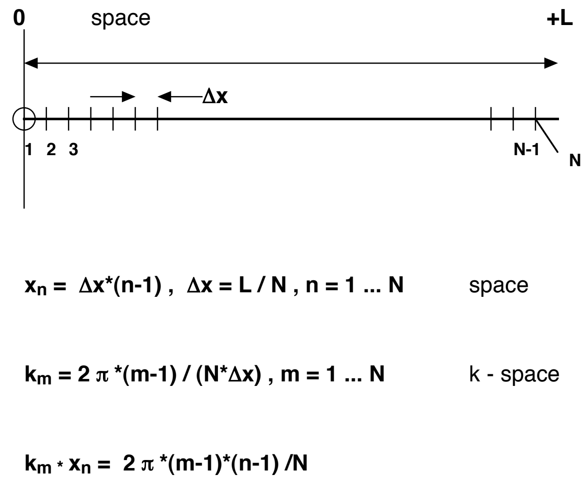

FIG. 1: x-space and k-space.

are also many standard manipulations of the elements in Fourier analysis in these numerical packages.

Sometimes you want to do something that nobody thought of before. Or you are ornary and just want to do it yourself. Then it is important to know how the computer stores Fourier components; you may want to pick them up and manipulate them. This is a *don't ask* situation. The N numbers $f_1 \cdots f_N$ are stored as a (column) vector as shown at the top of Fig. 2, i.e., from 1 to N. The component f_1 corresponds to $k_1 = 0$ and the component $f_{1+N/2}$ corresponds to $k_{1+N/2} = \pi$. All other Fourier components (for a real function) come in pairs with $f_{N-1} = f_2^*$, $f_{N-2} = f_3^*$, etc. The components f_2 $(f_3, f_4, \cdots)$and f_{N-1} $(f_{N-2}, f_{N-3}, \cdots)$ are corresponding *long wavelength* components of the Fourier representation of a function. The components $f_{N/2}$ $(f_{N/2-1}, \cdots)$and $f_{2+N/2}$ $(f_{3+N/2}, \cdots)$ are corresponding *short wavelength*

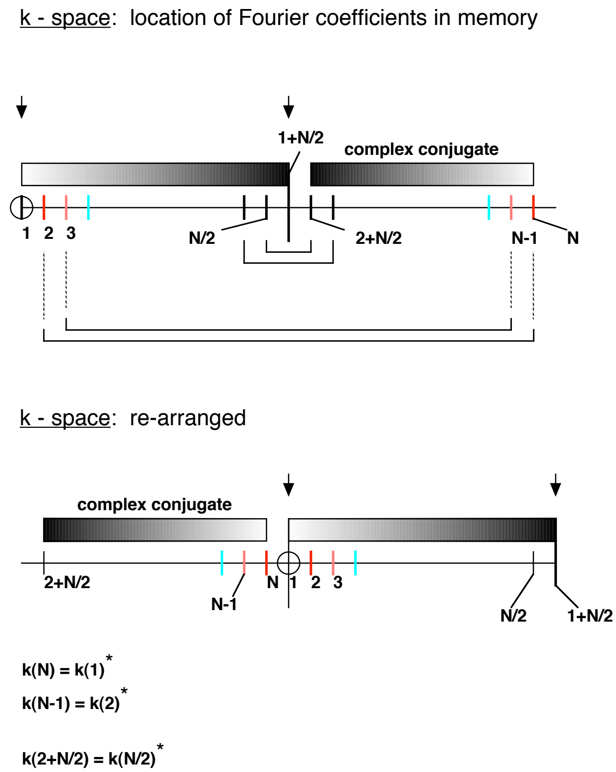

FIG. 2: Storage locations for FFT.

components of the Fourier representation of a function. They are usually thought of in the re-arranged form of $k - space$ at the bottom of Fig. 2. The question of where the Fourier components are located comes up for example when you might want to smooth a function (by removing the short wavelength components) or high pass filter a function by removing the long wavelength components. If you want to continue to have the function be *real* you have to take Fourier components away pairwise.

Nonlinearity. We will derive 2 (possibly more) nonlinear wave equations. For starters lets just look at a few such equations.

1. In the interior of a fluid (i.e., a fluid with no or unimportant free surface)

$$\frac{\partial^2 p}{\partial t^2} - c_0^2 \frac{\partial^2 p}{\partial x^2} = -\frac{\beta}{\rho_0 c_0^4} \frac{\partial^2}{\partial t^2}\ p^2, \tag{25}$$

where ρ_0 and c_0 are the density and sound velocity respectively and β is a parameter that measures the strength of the nonlinearity.

2. Korteweg-deVries equation (for almost shallow water waves)

$$\frac{\partial u}{\partial t} - c_0 \frac{\partial u}{\partial x} + \beta \frac{\partial^3 u}{\partial x^3} + \frac{\alpha}{2} \frac{\partial u^2}{\partial x} = 0, \tag{26}$$

where β is a measure of dispersion and α is a measure of nonlinearity. This is a *one way* wave equation.

3. The nonlinear Schroedinger equation (often encountered in the discussion of dilute bose gases and BEC)

$$-\frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2} - \alpha |\psi|^2 \psi = E\psi. \tag{27}$$

4. The ϕ^4 equation (encountered in a description of the *order parameter* and phase transitions)

$$\frac{\partial^2 \psi}{\partial x^2} = \alpha \psi - \beta \psi^3. \tag{28}$$

5. The sine-Gordon equation (encountered in a description of quantum fields, in a description of the order parameter in superfluid ^{3}He)

$$\frac{\partial^2 \psi}{\partial t^2} - c_0^2 \frac{\partial^2 \psi}{\partial x^2} = A \sin \psi. \tag{29}$$

Some of these equations are very similar to other of them. So while they describe different physical situations there is often a common mathematical scheme that can provide the solution. On the other hand be forewarned *for nonlinear problems there is a dearth of methods for solution compared to linear problems.*

A simplification of the ϕ^4 equation.

1. Divide by α and define $z = \sqrt{\alpha} x$ and $\gamma = \beta/\alpha$

$$\frac{\partial^2 \psi}{\partial z^2} = \psi - \gamma \psi^3. \tag{30}$$

2. Define $\phi = \lambda \psi$, where λ is a constant. Substitute and find that for $\lambda^2 = \gamma^{-1}$

$$\frac{\partial^2 \phi}{\partial z^2} = \phi - \phi^3. \tag{31}$$

A manipulation of the sine-Gordon equation.

1. Look for solution of the form $\psi = \phi(\zeta)$ with $\zeta = x - vt$. Then

$$\frac{\partial^2 \psi}{\partial t^2} - c_0^2 \frac{\partial^2 \psi}{\partial x^2} = (v^2 - c_0^2) \frac{\partial^2 \phi}{\partial \zeta^2}. \tag{32}$$

2. Define $z = \zeta \sqrt{A/(c_0^2 - v^2)}$ and find

$$\frac{\partial^2 \phi}{\partial z^2} = -\sin\phi = -\phi + \frac{1}{3!}\phi^3 + \cdots, \tag{33}$$

which is suspiciously like Eq. (31).

Solution to a nonlinear equation. Here is an illustration of a method of solution to the sine-Gordon equation in the approximation of Eq. (33). Before we start let's collect the ingredients

1. the equation

$$\frac{\partial^2 \psi}{\partial t^2} - c_0^2 \frac{\partial^2 \psi}{\partial x^2} = A \sin\psi, \tag{34}$$

2. look for solution in the form $\phi(\zeta = x - vt)$,

3. use variable $z = \zeta \sqrt{A/(c_0^2 - v^2)}$,

4. keep only the first two terms in $\sin\phi$,

5. scale ϕ by $\lambda = 1/\sqrt{3!}$, $\phi = \lambda X$

To solve

$$\ddot{X} = -X + X^3, \tag{35}$$

where $\dot{X}$ stands for dX/dz, multiply the equation by $\dot{X}$ and re-write

$$\dot{X}\ddot{X} = -\dot{X}X + \dot{X}X^3 \leftrightarrow \frac{d}{dz}\frac{\dot{X}^2}{2} = \frac{d}{dz}\left(-\frac{X^2}{2} + \frac{X^4}{4}\right). \tag{36}$$

Integration yields

$$\frac{\dot{X}^2}{2} = E - \frac{X^2}{2} + \frac{X^4}{4} = E - V(X), \tag{37}$$

where E is a constant of integration. Using the correspondence between this equation and

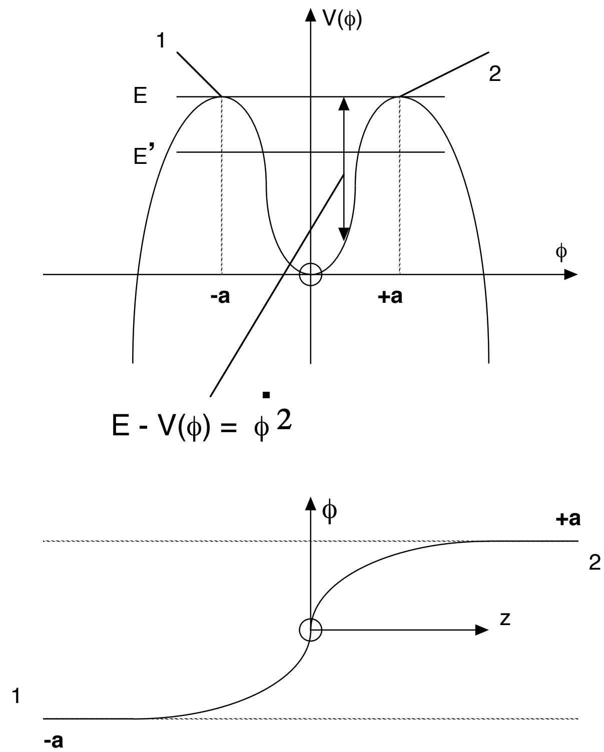

FIG. 3: Solution to the ϕ^4 equation by *quadrature* uses the correspondance of Eq. (37) to classical mechanics, $T = E - V$.

the energy picture in classical mechanics, Fig. 3, suggests that a particularly simple solution (certainly not the only solution) would follow from the choice $E = max(V(X)) = 1/4$. Then

$$\dot{X} = \frac{1}{\sqrt{2}}(1 - X^2). \tag{38}$$

Re-arrange

$$\frac{dX}{1 - X^2} = \frac{1}{\sqrt{2}} dz \tag{39}$$

and find

$$X = \tanh \frac{z}{\sqrt{2}}. \tag{40}$$

Now work back through the definitions to find ψ

$$\phi(x - vt) = \sqrt{3!} \tanh \frac{\kappa(x - vt)}{\sqrt{1 - \beta^2}}, \tag{41}$$

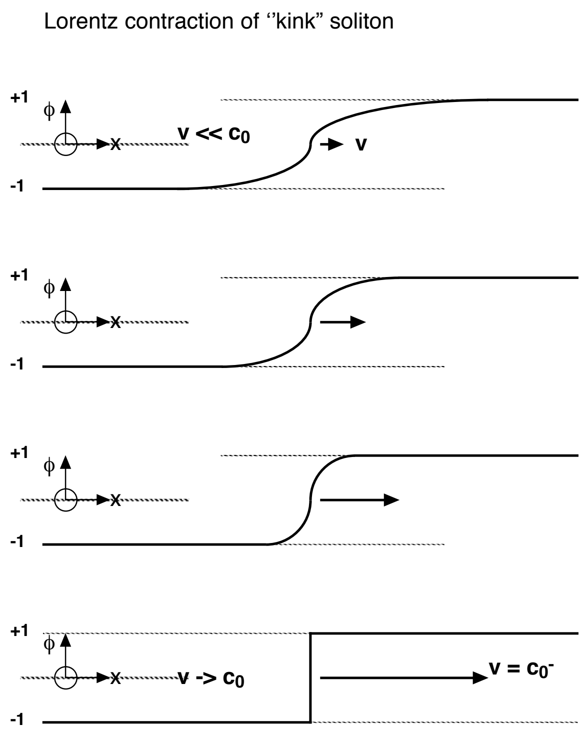

FIG. 4: Lorentz contraction of "kink" soliton.

where $\kappa = \sqrt{A/2}/c_0$ and $\beta = v/c_0$. This solution is a "'kink" that undergoes Lorentz contraction becoming a step at $v \rightarrow c_0$..

Aside. Model for a sine-Gordon like equation. A sine Gordon like equation describes a set of similar pendulums in gravity that are tied to a torsion fiber. The single particle force is $sin\theta$. The force due to twisting the torsion fiber is the the second difference

$$f_{torsion} = \tau(\theta_{n+1} - \theta_n) - \tau(\theta_n - \theta_{n-1}) \rightarrow \tau a^2 \frac{\partial^2 \theta}{\partial x^2}, \tag{42}$$

where a is the spacing between pendulums along the torsion fiber.

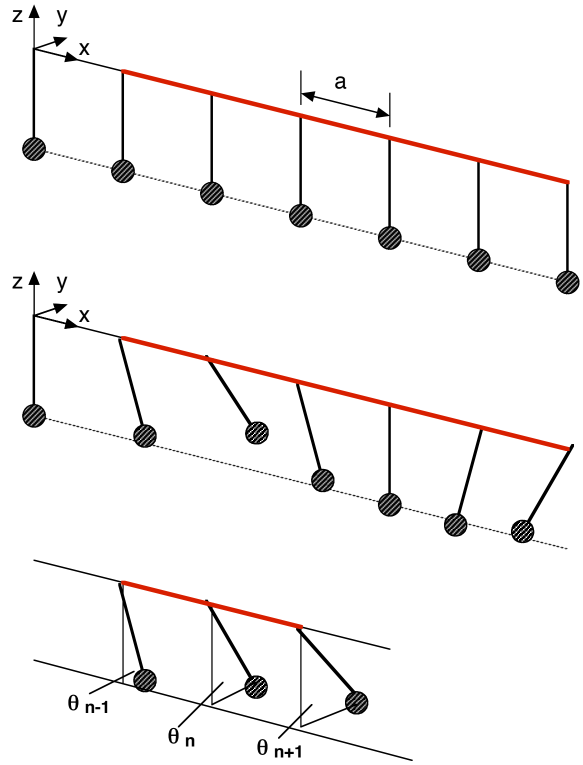

FIG. 5: Lorentz contraction of "kink" soliton.

Derive the Korteweg-deVries equation. This is going to be hard work. We want to consider water waves that are almost shallow. This means that we keep track of the importance of two small quantities, the amplitude of the disturbance, ζ, and kh_0, the ratio of the fluid depth to the wavelength of the disturbance. When $\zeta \to 0$ and $kh_0 \to 0$ we have Eqs. (1)-(4). A mild correction from this extreme is embodied in Eqs. (5)-(7). It is clear from the dispersion relation in Eq. (9) contains higher order terms in kh_0. We want to get at these terms in a systematic way. To start recall that the bounday conditions that were imposed on solution to Eq. (5) came from Bernoulli, which we linearized, and from $z = \zeta$. The Bernoulli equation, with the term nonlinear in v re-inserted, is

$$-\frac{\partial \phi}{\partial t} + \frac{1}{2}\nabla\phi \cdot \nabla\phi + g\zeta = 0. \tag{43}$$

From $z = \zeta$ we have

$$\frac{dz}{dt} = v_z = \frac{\partial \phi}{\partial z} = \frac{d\zeta}{dt} = \frac{\partial \zeta}{\partial t} + \nabla\phi \cdot \nabla\zeta, \tag{44}$$

or

$$\frac{\partial \phi}{\partial z} = \frac{\partial \zeta}{\partial t} + \nabla\phi \cdot \nabla\zeta, \tag{45}$$

When the nonlinear terms are dropped from Eqs. (43) and (45) we get Eq. (7).

To discover how kh_0 and ζ appear in ϕ consider a general disturbance of the fluid represented as a superposition of "plane waves" at $z = \zeta$

$$\phi(x, \zeta, t) = \int dk \;\cosh k(h_0 - \zeta)\; e^{ikx} A(k, t). \tag{46}$$

Develop in Taylor series

$$\begin{aligned}
\phi(x, \zeta, t) &= \int dk \;\left(1 + \frac{1}{2!}(k(h_0 - \zeta))^2\right)\; e^{ikx} A(k, t), && (47)\\
&= \int dk \;\left(1 + \frac{1}{2!}[h_0^2 - 2h_o\zeta + \zeta^2]k^2\right)\; e^{ikx} A(k, t), && (48)\\
&= \int dk \;\left(1 - \frac{1}{2!}[h_0^2 - 2h_o\zeta + \zeta^2]\frac{\partial^2}{\partial x^2}\right)\; e^{ikx} A(k, t), && (49)\\
&= \left(1 - \frac{1}{2!}[h_0^2 - 2h_o\zeta + \zeta^2]\frac{\partial^2}{\partial x^2}\right)\; A(x, t), && (50)
\end{aligned}$$

where

$$A(x, t) = \int dk \; e^{ikx} A(k, t). \tag{51}$$

Similarly

$$\begin{aligned}
\frac{\partial \phi}{\partial z} &= -\int dk \; k \sinh k(h_0 - \zeta)\; e^{ikx} A(k, t), && (52)\\
&= -\int dk \; k \left(k(h_0 - \zeta) + \frac{1}{3!}[k(h_0 - \zeta)]^3\right)\; e^{ikx} A(k, t), && (53)\\
&= (h_0 - \zeta)\frac{\partial^2 A(x, t)}{\partial x^2} - \frac{1}{3!}(h_0 - \zeta)^3 \frac{\partial^4 A(x, t)}{\partial x^4} && (54)
\end{aligned}$$

When these expansions for ϕ and $\partial\phi/\partial z$ are used in Eqs. (43) and (45) we obtain

$$g\zeta - \frac{\partial A}{\partial t} + \frac{h_0^2}{2}\frac{\partial^3 A}{\partial t \partial x^2} + \frac{1}{2}\left(\frac{\partial A}{\partial x}\right)^2 = 0 \tag{55}$$

and

$$\frac{\partial \zeta}{\partial t} = h_0 \frac{\partial^2 A}{\partial x^2} - \frac{1}{6} h_0^3 \frac{\partial^4 A}{\partial x^4} - \zeta \frac{\partial^2 A}{\partial x^2} - \frac{\partial A}{\partial x}\frac{\partial \zeta}{\partial x}. \tag{56}$$

1. The last term in Eq. (55) is nonlinear.

2. The last two terms in Eq. (56) are nonlinear.

3. Test these equations when the nonlinear terms are dropped. Write $\zeta = Zexp\, i(kx - \omega t)$ and $A = Bexp\; i(kx - \omega t)$ and find

$$-gZ \;=\; i\omega \left(1 - \frac{k^2 h_0^2}{2}\right) B, \tag{57}$$

$$-i\omega Z \;=\; -h_0 k^2 \left(1 - \frac{k^2 h_0^2}{6}\right) B. \tag{58}$$

These can be solved for ω^2 in agreement with the result in Eq. (11). The higher derivative terms in Eqs. (55) and (56) are a manifestation of dispersion.

Look for solutions to Eqs. (55) and (56) that are functions of $\eta = x - vt$. Then we have

$$0 \;=\; g\zeta + vA^{'} - \frac{vh_0^2}{2} A^{'''} + \frac{1}{2}(A^{'})^2, \tag{59}$$

$$0 \;=\; v\zeta^{'} + h_0 A^{''} - \frac{h_0^3}{6} A^{''''} - \zeta A^{''} - A^{'}\zeta^{'}, \tag{60}$$

where $y^{'} = dy/d\eta$. The second of these two equations is an exact derivative

$$A^{'} = -\frac{v}{h_0}\zeta + \frac{h_0^2}{6} A^{'''} + \frac{1}{h_0}\zeta A^{'} + constant. \tag{61}$$

We use $A^{'}$ in the first equation to find

$$\left(1 - \frac{gh_0}{v^2}\right)\zeta - \frac{3}{2h_0}\zeta^2 - \frac{h_0^2}{3}\zeta^{''} = 0. \tag{62}$$

This is one form of the KdV equation. To compare directly with Eq. (26), another form, use $u = u(x + vt)$ in Eq. (26) and remove one derivative from all terms.