P740.HW5.tex

Due 04/04/07

1. Shallow water A shallow channel has V-shaped cross section with breadth $2a$ and depth h_0 at the center, $x = 0$.

1. Show that the modes that flow back and forth across the channel (they are uniform along the channel) have even or odd symmetry about $x = 0$.

2. Find equations for the eigenvalues of both sets of modes. Find the lowest 6 eigenvalues.

3. Sketch the surface profile of the lowest 3 modes.

2. Bounded oil/water

1. Find the dispersion relation for the waves on the interface of 2 fluids. Fluid 1, density ρ_1, resides $0 \leq z \leq h_1$ above fluid 2, density $\rho_2 > \rho_1$, that resides in $-h_1 \leq z \leq 0$. Gravity is at work.

2. Consider various limits, e.g. $\rho_1 \to \rho_2$, etc.

3. Fourier transform

Find the Fourier transform of

$$S(t; t_0, \Delta t, \Omega) = S(0)\frac{1}{\sqrt{2\pi(\Delta t)^2}} e^{-\frac{1}{2}\frac{(t-t_0)^2}{(\Delta t)^2}} \sin \Omega(t - t_0). \tag{1}$$

4. Fourier transform and dispersion. At $t = 0$ the displacement of a shallow water wave is

$$\delta h_0(x) = Cexp(-x^2/(2W^2)), \tag{2}$$

where $W = 0.5$ and C is fixed by $\int \delta h(x)dx = 1$. Find the location and shape of this disturbance at later times. Assume that the dispersiion relation for water waves is (the almost shallow case)

$$\omega = c_0 k \left(1 - \alpha k^2\right), \tag{3}$$

where $\alpha = 0.0001$. The answer in principle is

$$\delta h(x, t) = \int \frac{dk}{2\pi}\ \delta h(k) e^{i(kx - \omega(k)t}, \tag{4}$$

where

$$\delta h(k) = \int dx \delta h(x) e^{-ikx}. \tag{5}$$

If there were no dispersion the solution to this problem would be $\delta h(x,t) = \delta h_0(x - c_0 t)$. With dispersion it is much harder. Re-arrange Eq. (4)

$$\delta h(x,t) = \int \frac{dk}{2\pi}\ \delta h(k) e^{-i\omega(k)t} e^{i(kx} = \int \frac{dk}{2\pi}\ \delta h(k,t) e^{i(kx}, \tag{6}$$

where

$$\delta h(k,t) = \delta h(k) e^{-i\omega(k)t}. \tag{7}$$

The solution, Eq. (6) is the inverse Fourier transform of $\delta h(k,t)$, a set of Fourier coefficients that evolve in time. Do this numerically (you may have to supply some parameters)

1. Choose the spatial interval over which you want to work.
2. Choose $\delta h(x)$. Norm it to 1 numerically.
3. Find $\delta h(k)$.
4. Make a choice of several moments of time, including $t = 0$. For each value of t form the factor $exp - i\omega(k)$ and $\delta h(k,t)$. [This is the place where this problem can be hard. Look at Note 12. Or ask.]
5. Find the inverse Fourier transform of $\delta h(k,t)$.
6. Plot $\delta h(x,t)$ vs x for a few times.
7. Comment on what you see.
8. Change the sign of α and repeat.

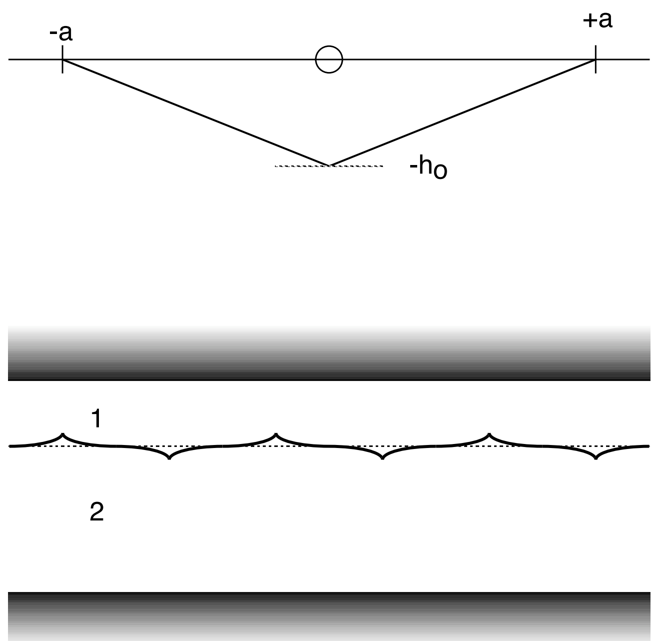

FIG. 1: