Midterm takehome exam, Fluid Dynamics

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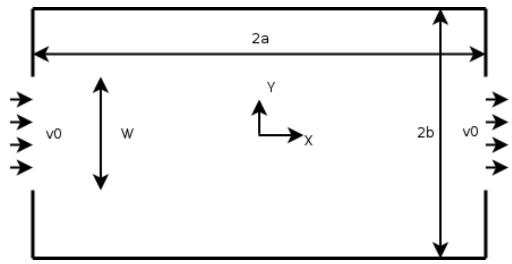

Figure 1: Two dimensional flow.

0.1 Problem #1

A steady two dimensional flow, as shown in Figure 1, enters and leaves a rectangular space, $2a \times 2b$, through identical openings. The flow is contained by the walls of the rectangular space. Choose w to be a fraction of 2b, $w = Q2b$.

0.1.1 Find the velocity potential for this flow

For this problem there is an axial symmetry. So we do not use any radial equations but we write for the flow

$$\begin{aligned} \Delta\phi &= 0, \\ \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} &= 0. \end{aligned}$$

Lets assume that solution for X component does not depend on solution for the Y component. Therefore we can write $\phi = X(x) \cdot Y(y)$, and the equation becomes

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} + \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} = 0.$$

So the terms of this equations must be equal and with opposite sign. As it ise reasonable to assume that cos term is describing the Y component of the equation and $sinh$ term is describing the X component of the equation, we can construct two equations

$$\begin{aligned} \frac{\partial^2 X}{\partial x^2} &= Xk^2, \\ \frac{\partial^2 Y}{\partial y^2} &= -Yk^2. \end{aligned}$$

By solving those, we get

$$\begin{aligned} X(x) &= A_X \cdot sinh(kx), \\ Y(y) &= B \cdot cos(ky) + C \cdot sin(ky). \end{aligned}$$

It is reasonable (and by rules) that we start with a boundary condtion, which has a value of zero. As the flow cannot enter to the horizontal walls, the derivative $\frac{dY}{dy}$ at the points $y = b, y = -b$ equals to zero. Also the flow in the upper half of the plane is antisymmetric to the flow in the lower half of the plane. So the sin function must describe the velocity of the flow. From that we can write

$$Y(y) = B \cdot cos(ky),$$

From the boundary conditions we'll get that

$$-Bk \cdot sin(kb) = 0,$$

so k must be $k = \frac{n\pi}{b}$. So now we can write ϕ in the form

$$\phi(x, y) = A \cdot sinh\left(\frac{\pi x}{b}\right) cos\left(\frac{n\pi y}{b}\right). \tag{1}$$

As we know the velocity of the flow at vertical boundaries, we must find the derivative of the potential, to match boundary conditions to the Eq. (1). At first, lets find the boundary condition. As the velocity is in the x direction and symmetrical with respect to x axis, we can describe the boundary velocity x component by using cosine series:

$$v(a, y) = v(-a, y) = \sum_{n=1}^{\infty} A_n cos\left(\frac{n\pi y}{b}\right) + A_0.$$

For $A_n$and A_0we can write:

$$\begin{aligned}
A_n &= \frac{2}{b}\int_0^b f(y) cos\left(\frac{n\pi y}{b}\right) dy \\
&= \frac{2}{b}\int_0^{w/2} v_0 cos\left(\frac{n\pi y}{b}\right) dy \\
&= \frac{2v_0}{n\pi} sin\left(n\pi Q\right) \\
A_0 &= \frac{1}{b}\int_0^b f(y) dy \\
&= \frac{1}{b} v_0 \frac{w}{2} \\
&= Q \cdot v_0
\end{aligned}$$

So the x directional flow on the boundaries is

$$v(a, y) = v(-a, y) = \sum_{n=1}^{\infty} \frac{2v_0}{n\pi} sin\left(n\pi Q\right) cos\left(\frac{n\pi y}{b}\right) + Qv_0. \tag{2}$$

Now we have Eq. (1) and Eq. (2) with one unknown variable A. To find that, we have to find derivative of ϕ on the vertical boundaries, where $x = a$ or $x = -a$.

$$\frac{\partial \phi}{\partial x} = \frac{A\pi}{b} cosh\left(\frac{\pi x}{b}\right) cos\left(\frac{n\pi y}{b}\right)$$

Next we'll find cosine series of the term $cosh\left(\frac{\pi x}{b}\right)=g(x)$:

$$g(x)=b_0+\sum_{n=1}^{\infty}b_n cos\left(\frac{n\pi x}{a}\right)$$

For b_n and $b_0$we'll get

$$\begin{aligned}
b_0 &= \frac{1}{a}\int_0^a cosh\left(\frac{\pi x}{b}\right)dx \\
&= \frac{b}{\pi a}sinh\left(\frac{\pi a}{b}\right) \\
b_n &= \frac{2}{a}\int_0^a cosh\left(\frac{\pi x}{b}\right)cos\left(\frac{n\pi x}{a}\right)dx \\
&= \frac{2}{a}\left(\frac{ab}{\left(a^2+b^2n^2\right)\pi}\right)\left(a\cdot cos\left(\frac{n\pi x}{a}\right)sinh\left(\frac{\pi x}{b}\right)\right)|_0^a \\
&= \frac{2ab\cdot cos\left(n\pi\right)sinh\left(\frac{\pi a}{b}\right)}{\left(a^2+b^2n^2\right)\pi} \qquad (3)
\end{aligned}$$

So for $\frac{\partial\phi}{\partial x}$ we can write now

$$\begin{aligned}
\frac{\partial\phi}{\partial x} &= \frac{A\pi}{b}\left[\frac{b}{\pi a}sinh\left(\frac{\pi a}{b}\right)+\sum_{n=1}^{\infty}b_n cos\left(\frac{n\pi x}{a}\right)\right]cos\left(\frac{n\pi y}{b}\right), \\
&= \sum_{n=1}^{\infty}\frac{A\pi}{b}b_n cos\left(\frac{n\pi x}{a}\right)cos\left(\frac{n\pi y}{b}\right)+\frac{A}{a}sinh\left(\frac{\pi a}{b}\right)cos\left(\frac{n\pi y}{b}\right).
\end{aligned}$$

By taking $n=0$ in the second term, we can simplify the equation a bit:

$$\frac{\partial\phi}{\partial x}=\sum_{n=1}^{\infty}\frac{A\pi}{b}b_n cos\left(\frac{n\pi x}{a}\right)cos\left(\frac{n\pi y}{b}\right)+\frac{A}{a}sinh\left(\frac{\pi a}{b}\right).$$

where $b_n$is given by equation (3). For boundary layers we can write:

$$\frac{\partial\phi}{\partial x}|_{x=-a,a}=\sum_{n=1}^{\infty}\frac{b_n A\pi}{b}cos\left(n\pi\right)cos\left(\frac{n\pi y}{b}\right)+\frac{A}{a}sinh\left(\frac{\pi a}{b}\right). \qquad (4)$$

By comparing this result to the Eq. (2), we can get the value of A (using $x=a$):

$$\begin{aligned}
\sum_{n=1}^{\infty}\frac{b_n A\pi}{b}cos\left(n\pi\right)cos\left(\frac{n\pi y}{b}\right) &= \sum_{n=1}^{\infty}\frac{2v_0}{n\pi}sin\left(n\pi Q\right)cos\left(\frac{n\pi y}{b}\right) \\
\frac{b_n A\pi}{b}cos\left(n\pi\right) &= \frac{2v_0}{n\pi}sin\left(n\pi Q\right) \\
\frac{2a\cdot cos^2\left(n\pi\right)sinh\left(\frac{\pi a}{b}\right)A}{\left(a^2+b^2n^2\right)} &= \frac{2v_0}{n\pi}sin\left(n\pi Q\right)
\end{aligned}$$

By using the fact that $cos^2(n\pi) = 1$, we can write for A:

$$A = \frac{v_0}{an\pi \cdot sinh\left(\frac{\pi a}{b}\right)} sin\left(n\pi Q\right)\left(a^2 + b^2 n^2\right). \tag{5}$$

Lets test the result. If A is really as shown by the Eq. (5), then the second terms in Eq. (4) and (2) must be equal!

$$\begin{aligned} \frac{A}{a} sinh\left(\frac{\pi a}{b}\right) &= Qv_0 \\ \frac{v_0}{a^2 n\pi} sin\left(n\pi Q\right)\left(a^2 + b^2 n^2\right) &= Qv_0. \end{aligned}$$

To show that, let's assume that $n = 0$ for the term A:

$$\frac{v_0}{a^2 n\pi} sin\left(n\pi Q\right)\left(a^2 + b^2 n^2\right) \to v_0 Q \frac{sin\left(n\pi Q\right)}{n\pi Q} = v_0 Q.$$

So that's really the case. So we can write for A instead:

$$A = \frac{Qv_0 a}{sinh\left(\frac{\pi a}{b}\right)}. \tag{6}$$

We could have done this at first place, but now the validity of the result has also been proven. By replacing this result, Eq. (6), to the Eq. (1), and also considering the second term in $\frac{\partial \phi}{\partial x}$ equation, we'll get the equation for ϕ:

$$\phi(x, y) = \frac{Qv_0 a}{sinh\left(\frac{\pi a}{b}\right)} \cdot sinh\left(\frac{\pi x}{b}\right) cos\left(\frac{\pi y}{b}\right) + Qv_0 x. \tag{7}$$

0.1.2 Make study of $v_x(0,0)/v_0$as function of a

The b value is fixed.

$$v_x = \frac{\partial \phi}{\partial x} = \frac{Qv_0 a\pi}{sinh\left(\pi \frac{a}{b}\right) b} cosh\left(\frac{\pi x}{b}\right) cos\left(\frac{\pi y}{b}\right) + Qv_0, \tag{8}$$

Considering only values $x = 0$ and $y = 0$

$$\frac{v_x\left(0,0\right)}{v_0} = \frac{Qa\pi}{sinh\left(\pi \frac{a}{b}\right) b} + Q. \tag{9}$$

Physically should be reasonable to assume that when the a is really big in comparsion to b, the flow must be $Q \cdot v_0$ and if $Q = 1$, the flow is $v_0$everywhere. Also when a is really small compared to b, the flow should be equal to $v_0$because streamlines do not bend with a really small a. As we can see, it really is true for Eq. (9). in Figure 2

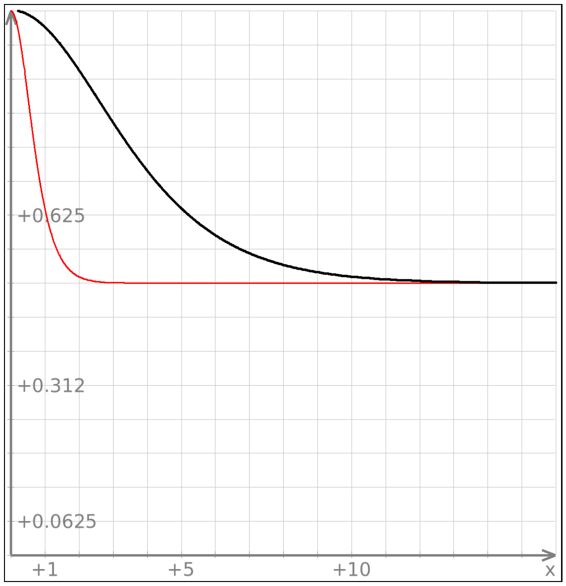

Figure 2: X directional velocity of the flow at the point $(0,0)$ for $b=5$ and $Q=0.5$ is denoted by red (thin) line. X axis is a for this case. Dark (bold) line shows $v_x(0,0)/v_0$ dependence on r for same Q value. X axis is r for this case.

0.1.3 Show that $v_x(0,0)/v_0$ is a function of Q and r

r is defined as $r=\frac{a}{b}$. From the Eq. (9) it is easy to see that the claim in the title of the subsection is really the case:

$$\frac{v_x\left(0,0\right)}{v_0}=Q\left(\frac{\pi r}{sinh\left(\pi r\right)}+1\right).$$

The result is shown in Figure 2, denoted by the bold line.

0.1.4 Pressure in the fluid along $y=0$.

Lets calculate the pressure in the fluid as a function of x along y=0 line. It is obvious to estimate that pressure follows the shape of cosh function, because this describes the velocity magnitude in x direction. Also we can assume that there exists a streamline, which follows the x axis (at y=0). That's because of the symmetry of the problem. Considering that and Eq. (8), the $v_x(x,0)$ is:

$$v_x(x,0)=\frac{Qv_0r\pi}{sinh\left(\pi r\right)}cosh\left(\frac{\pi x}{b}\right)+Qv_0.$$

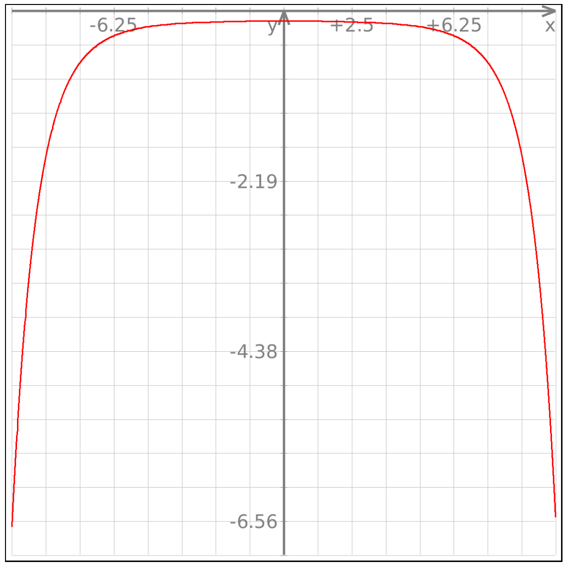

Figure 3: The shape of the pressure in the tube, where $a = 2b = 10$ and $P_0 = 0$.

Using Bernoull'i equation for the streamline, we the following result:

$$\begin{aligned}
\frac{1}{2}v_0^2 + \frac{P_0}{\rho} &= \frac{1}{2}v_x^2(x,0) + \frac{P(x)}{\rho} \\
P(x) &= \frac{\rho v_0^2}{2} + P_0 - \frac{\rho v_x^2(x,0)}{2} \\
&= C - \frac{\rho}{2}\left(\frac{Qv_0 r\pi}{sinh\left(\pi r\right)}cosh\left(\frac{\pi x}{b}\right) + Qv_0\right)^2.
\end{aligned}$$

By replacing constants with some numbers, we'll get the result as shown in Figure 3.

0.1.5 Streamline sketch

For $Q = 1/3$ and $a = 3b$, the streamlines in 2D could be seen in Figure 4 and streamlines, potential lines and velocity magnitude could be seen in Figure 5.

0.2 Problem #2

Description: An ocean is bounded near the shore by a bottom, that fals off with x. Thus $h_0(x) = h_0 x/a$, where a is a length. Consider the shallow water waves on this ocean.

Figure 4: Streamlines in 2D. Flow enters from the right side and leaves from the left side.

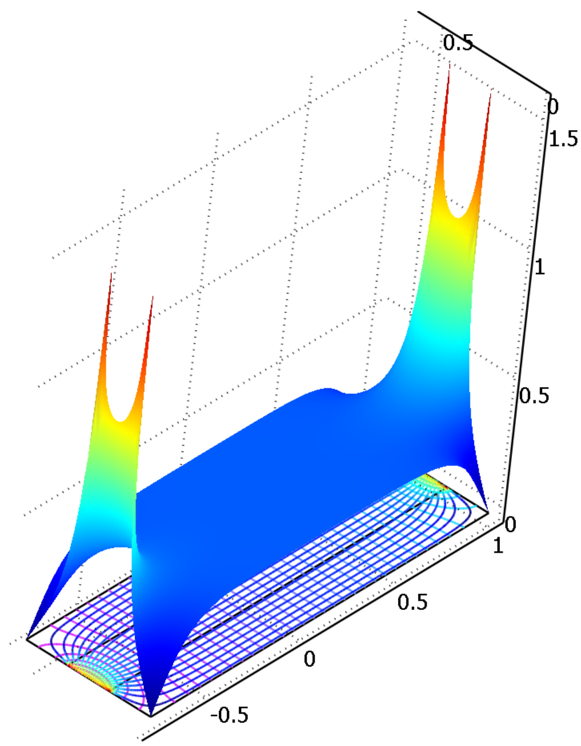

Figure 5: On the bottom, there are potential lines and streamlines. 3D part shows the magnitude of the velocity, i.e. $|grad\phi|$.

0.2.1 What wave equation describes $\delta h(x,t)$.

$\delta h(x,t)$ denotes surface fluctuations from h_0, where $h_0(x) = h_0 x/a$. To get a wave equation, we have to write down 2 equations - fluctuations in time and in space. Let's begin with time dependence and consider a region of fluid, which is really small in x direction. Equation of continuity states that the amount of fluid flowing into the region must be equal to the amount of fluid flowing out of the region. So if from the left side flows in more fluid than flows out from the right side, there must occur the change of the height of the fluid, as the fluid is incompressible. For that situation we can write down the equation:

$$\begin{aligned}
\rho_0 h_0 \left(x - \frac{dx}{2}\right) Lv \left(x - \frac{dx}{2}\right) - \rho_0 h_0 \left(x + \frac{dx}{2}\right) Lv \left(x + \frac{dx}{2}\right) &= \rho_0 dx L \frac{\partial \delta h}{\partial t}, \\
\frac{\partial \delta h}{\partial t} + h_0 \frac{\partial}{\partial x}\left(\frac{x}{a} v(x)\right) &= 0, \\
\frac{\partial \delta h}{\partial t} + h_0 \frac{v(x)}{a} + h_0 \frac{x}{a}\frac{\partial v(x)}{\partial x} &= 0. \qquad (10)
\end{aligned}$$

The next equation could be found from Euler equation for incompressible fluids:

$$\frac{\partial v}{\partial t} = -\frac{1}{\rho_0}\nabla P.$$

We are interested in wave movement in x direction only, because $\frac{\partial v_z}{\partial t} \approx 0$. So for z component of the gradient we can write

$$\frac{\partial P}{\partial z} = \rho_0 g \rightarrow P = P_0 + \rho_0 g\left(\delta h(x) - z\right).$$

For the v_x the equation

$$\frac{\partial v_x}{\partial t} = -\frac{1}{\rho_0}\rho_0 g \frac{\partial \delta h(x)}{\partial x} \Rightarrow \frac{\partial v_x}{\partial t} + g\frac{\partial \delta h(x)}{\partial x} = 0 \qquad (11)$$

must hold. So, by multiplying Eq. (10) by $\frac{\partial}{\partial t}$ and Eq. (11) by $\frac{\partial}{\partial x}$, we are able to obtain a wave equation for δh:

$$\begin{aligned}
\frac{\partial^2 \delta h}{\partial t^2} + \frac{h_0}{a}\left[\frac{\partial v(x)}{\partial t} + x\frac{\partial^2 v(x)}{\partial t \partial x}\right] &= 0, \qquad (12) \\
\frac{\partial v_x^2}{\partial x \partial t} + g\frac{\partial^2 \delta h(x)}{\partial x^2} &= 0. \qquad (13)
\end{aligned}$$

By substituting first term of Eq. (13) and Eq. (11) into the Eq. (12), the wave equation gets the form:

$$\frac{\partial^2 \delta h}{\partial t^2} - \frac{g h_0}{a}\left[\frac{\partial \delta h}{\partial x} + x\frac{\partial^2 \delta h}{\partial x^2}\right] = 0. \qquad (14)$$

0.2.2 Find the steady state solution to the equation (14).

Find the steady state solution to the Eq. (14), i.e. solution δh of the form $\delta h(x,t) = H(x)cos\omega t$. The ODE could be sanitized by using $z = x/a$, $k^2 = \omega^2 a^2/c_0^2$ and $c_0^2 = gh_0$.

What we are trying to do next is to manipulate the equation to get it to some common form which could be solved rather easily. At first the equation is multiplied by x^2, which results in the form of

$$x^2\frac{\partial^2\delta h}{\partial t^2} - \frac{gh_0}{a}x^3\frac{\partial^2\delta h}{\partial x^2} - \frac{gh_0}{a}x^2\frac{\partial\delta h}{\partial x} = 0.$$

Next we replace δh by $H(x)cos\omega t$ and also divide by $-cos\omega t$:

$$x^2\omega^2 H(x) + \frac{gh_0}{a}x^3\frac{\partial^2 H(x)}{\partial x^2} + \frac{gh_0}{a}x^2\frac{\partial H(x)}{\partial x} = 0.$$

Now we have an ordinary differential equation. By replacing z and c_0 into the equation, we will get little bit nicer form:

$$\begin{aligned} c_0^2 z x^2\frac{\partial^2 H(x)}{\partial x^2} + c_0^2 z x\frac{\partial H(x)}{\partial x} + x^2\omega^2 H(x) &= 0, \\ x^2\frac{\partial^2 H(x)}{\partial x^2} + x\frac{\partial H(x)}{\partial x} + \frac{\omega^2 a}{c_0^2}xH(x) &= 0. \end{aligned} \tag{15}$$

According to Wolfram Mathworld (http://mathworld.wolfram.com/BesselDifferentialEquation.html), the obtained Eq. (15) is a transformed form of the Bessel differential equation in the form:

$$x^2\frac{d^2H}{dx^2} + (2p+1)\,x\frac{dH}{dx} + \left(\alpha^2 x^{2r} + \beta^2\right)H = 0,$$

with a solution

$$H = x^{-p}\left[C_1 J_{q/r}\left(\frac{\alpha}{r}x^r\right) + C_2 Y_{q/r}\left(\frac{\alpha}{r}x^r\right)\right],$$

where $q \equiv \sqrt{p^2-\beta^2}$. It is easy to see, that for Eq. (15), $p = 0$, $\beta = 0$, $r = \frac{1}{2}$ and $\alpha = \frac{\omega\sqrt{a}}{c_0} = \frac{k}{\sqrt{a}}$. So a solution for the equation would be

$$H(x) = C_1 J_0\left(\frac{2k}{\sqrt{a}}\sqrt{x}\right) + C_2 Y_0\left(\frac{2k}{\sqrt{a}}\sqrt{x}\right), \tag{16}$$

which is a steady state solution for the Eq. (14).

0.2.3 Fix ω and make a plot of $H(x)$.

If we assume, that wavelength of a shallow water wave is quite big... lets say about 100m, then k is about 0.01. But we can assume also that a is not a really large number, rather less than 1. So I would think that we could get

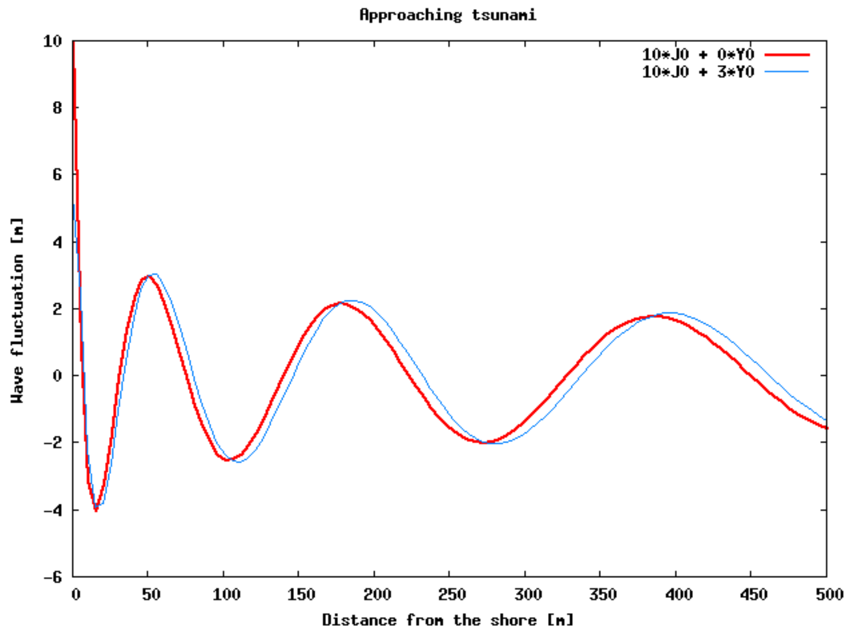

Figure 6: Two solutions for the Bessel Eq. (15)

quite realistic results assuming that $\frac{2k}{\sqrt{a}} \approx 1$. By fixing that, we should also consider values of $C_1$and C_2. As known, Y_0 gets to minus infinity around zero. So it would be rather unphysical, because wave amplitude cannot get "into the ground" near the shore. So, if $Y_0$is to be considered at all, then the C_2 should be much less than C_1. Then the Y_0 would be a term, which decreases the height of the wave near the shore. Physically, it could mean for instance dispersion factor for the big amplitude. The both solution could be seen in Figure 6 and they are both physically valid. It means that by approaching the shore, the amplitude of the tsunami increases and the wavelength decreases.

0.3 Problem #3

There is a small circular hole, cross sectional area A_0, in the bottom of the tank in Figure (7), filled to height of h with water. What is the cross section of the fluid flow at points $z > 0$ below bottom of the tank, i.e., $A(z)$.

We can use Bernoulli equation for the problem - Bernoulli equation could be solved on a streamline beginning on the top of the water level and continuing through the hole in the bottom of the tank. We can assume that the tank is large enough that water level at the height h does not change with the flow. So

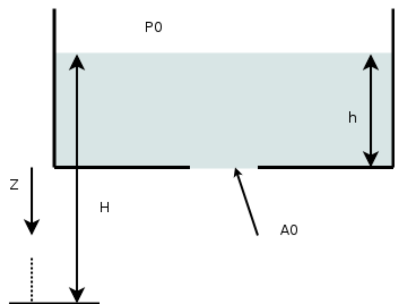

Figure 7: The tank of fluid.

the Bernoulli equation for the given system could be written as

$$\frac{P_0}{\rho} + Hg = \frac{1}{2}v^2 + \frac{P_0}{\rho} + g\left(H - h - z\right). \tag{17}$$

On the both side of the equation, there is a term $\frac{P_0}{\rho}$. We could write this, because pressure on the fluid is P_0, when $z > 0$ or $z = -h$. So Eq. (17) is good for $z > 0$. The variable H is a distance from the ground and $H \gg z$. It means H is somewhere far away from the tank. This variable must be considered because of the change of the potential energy along the the streamline in the flowing fluid. After manipulating the given equation, we can get $v(z)$:

$$\begin{aligned} \frac{1}{2}v(z)^2 &= g\left(h + z\right), \\ v(z) &= \sqrt{2g\left(h+z\right)}. \end{aligned} \tag{18}$$

So now we have fluid velocity dependence on z, whereas $z > 0$. By assuming, that the flow is continuous, we can claim that flux of fluid flow during some time period Δt must be constant throughout the flow. Mathematically it would look like

$$A_0 v(0)\Delta t = A(z)v(z)\Delta t.$$

So from this relation, we can get the cross section of the fluid:

$$A(z) = \frac{A_0 v(0)}{v(z)} = \frac{A_0\sqrt{2gh}}{\sqrt{2g\left(h+z\right)}} = \frac{A_0\sqrt{h}}{\sqrt{h+z}}. \tag{19}$$

0.4 Problem #4

A point source of fluid, with strength μ, is located at (a,b) near two perpendicular walls. Nearby the source the flow is radially outward with velocity v_0. To solve the problem, we introduce 3 image charges which should guarantee, that there is no flux across any boundary of the system. See Figure 8.

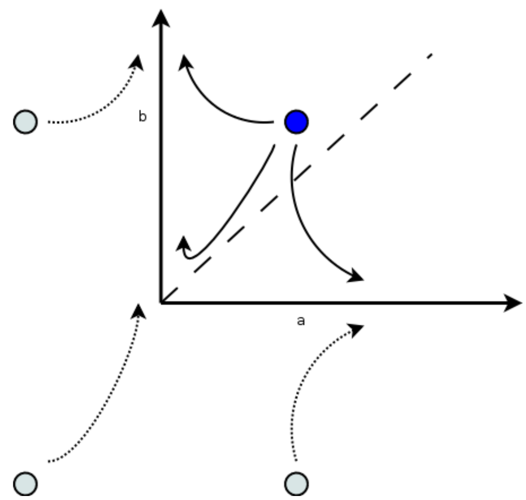

Figure 8: A charge and corresponding image charges. Also empirically estimated flow.

0.4.1 Find $\vec{v}$ for this flow

So as potential of a velocity of a flow is described by real part of the $F(z)$ and if $F(z) = ln(z)$, the flow is radially flowing out of point $(0,0)$. So for four charges, which has been shifted to different positions in the plane, we can write equation for ϕ

$$\begin{aligned}\phi(x,y) &= \mu\left(ln\sqrt{(x-a)^2+(y-b)^2}+ln\sqrt{(x+a)^2+(y-b)^2}\right)+ \\ &+ \mu\left(ln\sqrt{(x+a)^2+(y+b)^2}+ln\sqrt{(x-a)^2+(y+b)^2}\right) \qquad (20)\end{aligned}$$

The equation is quite ugly actually. But due to the 2 dimensions and 4 charges, it seems to be only possible way to describe the system. Other way would be to use the Green function, but it could turn out to be much more difficult. So let's find velocity vector components:

$$\begin{aligned}v_x &= \mu\left[\frac{(x-a)}{(x-a)^2+(y-b)^2}+\frac{(x+a)}{(x+a)^2+(y-b)^2}\right]+ \\ &+ \mu\left[\frac{(x+a)}{(x+a)^2+(y+b)^2}+\frac{(x-a)}{(x-a)^2+(y+b)^2}\right] \qquad (21)\end{aligned}$$

$$\begin{aligned}v_y &= \mu\left[\frac{(y-b)}{(x-a)^2+(y-b)^2}+\frac{(y-b)}{(x+a)^2+(y-b)^2}\right]+ \\ &+ \mu\left[\frac{(y+b)}{(x+a)^2+(y+b)^2}+\frac{(y+b)}{(x-a)^2+(y+b)^2}\right] \qquad (22)\end{aligned}$$

It is easy to see, that boundary condition are satisfied for those equations. If $x = 0$, v_x must be also 0:

$$v_x(x=0,y) = \mu \left[\frac{-2a}{a^2 + (y-b)^2} + \frac{2a}{a^2 + (y-b)^2} \right] = 0$$

Similarly is for v_y. If $y = 0$, the v_y must be also 0. So we have found $\vec{v}$ for the flow, with compoents described in equations (21) and (22).

0.4.2 Find $v_x$along the wall at $y = 0$.

How does the velocity vary with large x also? I $y = 0$, there is only x component of $\vec{v}$. Using Eq. (21). we can write for v_x

$$v_x(x, y=0) = \mu \left[\frac{2(x-a)}{(x-a)^2 + b^2} + \frac{2(x+a)}{(x+a)^2 + b^2} \right] \tag{23}$$

If $x = 0$, the v_x is also zero. If x is very big, the $v_x \to 0$ also. The shape of the graph for values $\mu = 0$, $a = 4$ and $b = 5$ could be seen in Figure 9. If x is big, we can neglet the term b^2 and a, then

$$v_x(x \gg 0, y=0) = 4\mu \left(\frac{1}{x} \right)$$

So v_x is inversely proportional to x for large values of x.

0.4.3 Now, a=b

0.4.3.1 Pressure variation along $y = 0$ wall.

We can use Eq. (23), to calculate velocity along the wall $y = 0$:

$$v_x(x, y=0) = \mu \left[\frac{2(x-a)}{(x-a)^2 + a^2} + \frac{2(x+a)}{(x+a)^2 + a^2} \right]$$

As there is no y component of velocity along the wall, we could use Bernoull'i equation, which is constant along a streamline. At $(0,0)$, the velocity is 0. Correspondingly, we can write the equation

$$\begin{aligned} \frac{P_0}{\rho} &= \frac{1}{2} v_x^2(x) + \frac{P(x)}{\rho} \\ P(x) &= P_0 - \frac{1}{2} \rho v_x^2(x) \\ P(x) &= P_0 - 2\rho\mu^2 \left[\frac{2(x-a)}{(x-a)^2 + a^2} + \frac{2(x+a)}{(x+a)^2 + a^2} \right]^2 \end{aligned} \tag{24}$$

The shape of the $P(x)$ could also be seen in Figure 9 (red line). The pressure is maximum at the point $(0,0)$, because $v_x$is zero there. Everywhere else the pressure is lower and it is lowest at the point where the velocity is maximum.

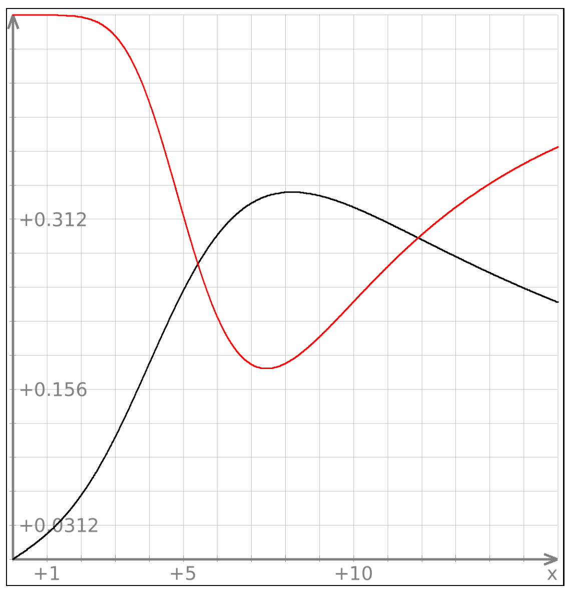

Figure 9: Blue line: $v_x$at y=0 and for values $\mu = 0$, $a = 4$ and $b = 5$. The red line is pressure variation for the case $a = b = 4$ with other parameters $P_0 = 0.5$ and $\rho = 1$. Notice that these graphs are for different sources. Blue is for the source, locating asymmetrically. Red is for the fluid source located at (a, a).

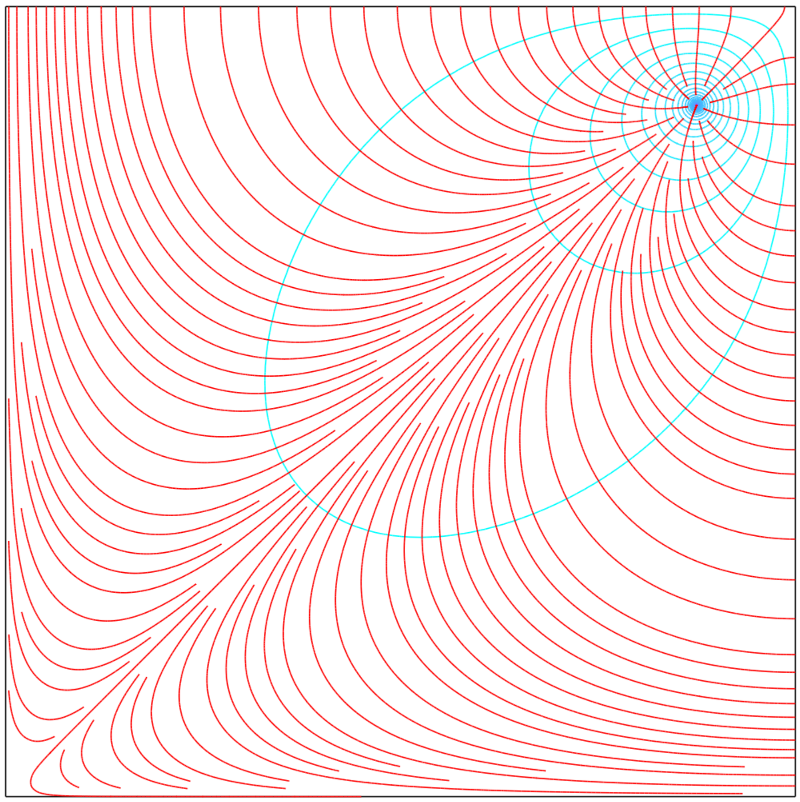

Figure 10: The uniform density streamlines caused by a point source.

0.4.3.2 Find v_x along the y=x

How does $v_x$change for large x. For this problem, we can use Eq. (21):

$$\begin{aligned} v_x(y=x)_{b=a} &= \mu\left[\frac{1}{2(x-a)}+\frac{(x+a)}{(x+a)^2+(x-a)^2}+\frac{1}{2(x+a)}+\frac{(x-a)}{(x-a)^2+(x+a)^2}\right] \\ &= \mu\left[\frac{x}{x^2-a^2}+\frac{x}{x^2+a^2}\right] \end{aligned} \tag{25}$$

Again, for large x, the velocity is inversely proportional to x. $v_x(x \gg a) \approx 2\mu\left(\frac{1}{x}\right)$ It is similar to the result found for the wall $y=0$. Only difference is in the constant coefficient.

0.4.3.3 Plot of streamlines

The plot of streamlines is in Figure 10.

0.5 Problem #5

Problem description: A plane with very long wing span (U2) is designed to create a constant amout of circulation Γ regardless of the forward speed. This airplane is approximated as a cylinder of radius a, length L and mass M around which there is a constant circulation.

0.5.1 The vertical force F

What is the vertical force F on the plane when it is moving at a horizontal speed of U (relative to the ground) in unbounded fluid, i.e., far from any surfaces? The radial velocity compoent is 0 on the surface of the cylinder. The angular component is (as derived in the class):

$$v_\theta = -\frac{u}{a^2}\left(1+\frac{a^2}{r^2}\right) sin\left(\theta\right) - \frac{\Gamma a}{2\pi r} = -u\left(\frac{2}{a^2} sin\left(\theta\right) + \frac{\Gamma}{2\pi u}\right).$$

Along a streamline around the cylinder the Bernoulli equations holds true:

$$\frac{1}{2}v^2\left(\theta\right) + \frac{P_0}{\rho} = const.$$

The pressure exerts force on to the each point of the cylinder surface $F = -P \cdot dS$ (- causes force to point out of the surface). As we are only interested in lift forces (and there are now other forces due to the symmetry), the y direction of the force is $F_y = -P \cdot dS \cdot sin\left(\theta\right)$. From Bernoulli equations the pressure is

$$P\left(\theta\right) = C\rho_0 - \frac{\rho_0}{2} v^2\left(\theta\right)$$

So differential component of the vertical force could be written as

$$dF_y = -L \cdot a \cdot d\theta \cdot P\left(\theta\right) \cdot sin\left(\theta\right).$$

Replacing $P(\theta)$ and v^2, we'll get for dF_y

$$dF_y = -La\left(C\rho_0 sin\left(\theta\right) - \frac{\rho_0}{2}u^2\left(\frac{4}{a^2}\cdot sin^3\left(\theta\right) + \frac{4}{a^2} sin^2\left(\theta\right)\frac{\Gamma}{2\pi u} + sin\left(\theta\right)\left(\frac{\Gamma}{2\pi u}\right)^2\right)\right) d\theta$$

It is easy to see that sin terms with an odd power are subject to disappear in integration (they cancel out in opposite sides of the cylinder).Only term to consider while integrating is the term with $sin^2\left(\theta\right)$. So the vertical net force is

$$\begin{aligned} F_y &= \frac{\Gamma a L \rho_0 u^2}{\pi u a^2}\int_0^{2\pi} sin^2\left(\theta\right) d\theta \\ &= \frac{\Gamma L \rho_0 u}{2\pi a}\left[\theta - cos\left(\theta\right) sin\left(\theta\right)\right]_0^{2\pi} \\ &= \frac{\Gamma L \rho_0 u}{a} \end{aligned} \tag{26}$$

0.5.2 Support against gravity

For what value of U is the plane supported against gravity? The gravity force on the plane is $F_g = -Mg$. So we'll have to solve the equation

$$-Mg + \Gamma\frac{L}{a}\rho_0 u = 0$$

and the plane is supported on velocity of at least with the value:

$$u = \frac{Mga}{\Gamma L \rho_0}.$$

0.5.3 Distance h from the ground

When the plane flies at horizontal velocity u (relative to the ground) at a distance h above the ground, $a \ll h$, what vertical force F will the plane generate? The basis for solving this problem is the following equation:

$$\phi = \frac{u}{a}\left(\frac{a}{r} + \frac{r}{a}\right) cos\left(\theta\right) - \frac{\Gamma a}{2\pi}\theta.$$

This equation describes a flow and circulation around a spherical object. To model distance from the ground, we have to add some extra terms. For that we use image "plane". So in our configuration one plane is located h above the ground and other one is located distance h below the ground. The modified equation for ϕ for that case is

$$\begin{aligned}
\phi &= \frac{u}{a}\left(\frac{a}{\sqrt{x^2+(y-h)^2}} + \frac{\sqrt{x^2+(y-h)^2}}{a}\right)\frac{x}{\sqrt{x^2+(y-h)^2}} + \\
&+ \frac{u}{a}\left(\frac{a}{\sqrt{x^2+(y+h)^2}} + \frac{\sqrt{x^2+(y+h)^2}}{a}\right)\frac{x}{\sqrt{x^2+(y+h)^2}} + \\
&+ \frac{\Gamma a}{2\pi}\left(-atan\left(\frac{y-h}{x}\right) + atan\left(\frac{y+h}{x}\right)\right).
\end{aligned}$$

This equation describes both planes and circulation. As we know there is now radial component around the wings of the planes, so what we want to do is manipulate this equation to get it dependent on θ and r only, where $r = a$. For the first two, noncircular terms we get, by replacing $\sqrt{x^2+y^2} = r = a$ and $x = r \cdot cos(\theta)$ and $y = r \cdot sin(\theta)$:

$$\begin{aligned}
\phi_u &= \frac{u}{a}\left(\frac{2x}{a} + \frac{ax}{x^2+(y-h)^2} + \frac{ax}{x^2+(y+h)^2}\right) \\
&= u \cdot cos\left(\theta\right)\left(\frac{2}{a} + \frac{a}{r^2+h^2-2hr\cdot sin\left(\theta\right)} + \frac{a}{r^2+h^2+2hr\cdot sin\left(\theta\right)}\right) \\
&\approx u \cdot cos\left(\theta\right)\left(\frac{2}{a} + \frac{2a}{h^2-4a^2 sin^2(\theta)}\right) \approx u \cdot cos\left(\theta\right)\left(\frac{2}{a} + \frac{2}{ah^2}\right).
\end{aligned}$$

We got this result by getting rid of r^2in the sum $\left(r^2+h^2\right)$, because h^2 is much bigger. Also we can see that if h get really large, the second term vanishes and we would have exactly the same result as for unbounded flow. Now let's find ϕ_Γ:

$$\begin{aligned}
\phi_\Gamma &= \frac{\Gamma a}{2\pi}\left(-atan\left(\frac{y-h}{x}\right)+atan\left(\frac{y+h}{x}\right)\right)\\
&= \frac{\Gamma a}{2\pi}\left(atan\left(\frac{r\cdot sin\left(\theta\right)+h}{r\cdot cos\left(\theta\right)}\right)-atan\left(\frac{r\cdot sin\left(\theta\right)-h}{r\cdot cos\left(\theta\right)}\right)\right).
\end{aligned}$$

Now let's take derivatives with respect to θ. For $\phi_u$we'll get:

$$\frac{\partial\phi_u}{\partial\theta}\approx-2\frac{u}{a^2}sin\left(\theta\right)\left(\frac{1}{h^2}+1\right).$$

For ϕ_Γ:

$$\begin{aligned}
\frac{\partial\phi_\Gamma}{\partial\theta} &= \frac{\Gamma}{2\pi}\left(\frac{r^2cos^2\left(\theta\right)+r^2\cdot sin^2\left(\theta\right)+hr\cdot sin\left(\theta\right)}{\left[1+\left(\frac{r\cdot sin(\theta)+h}{r\cdot cos(\theta)}\right)^2\right]r^2cos^2\left(\theta\right)}\right)-\\
&- \frac{\Gamma}{2\pi r}\left(\frac{r^2cos^2\left(\theta\right)+r^2\cdot sin^2\left(\theta\right)-hr\cdot sin\left(\theta\right)}{\left[1+\left(\frac{r\cdot sin(\theta)-h}{r\cdot cos(\theta)}\right)^2\right]r^2cos^2\left(\theta\right)}\right)=\\
&= \frac{\Gamma}{2\pi}\left(\frac{r^2+hr\cdot sin\left(\theta\right)}{h^2}-\frac{r^2-hr\cdot sin\left(\theta\right)}{h^2}\right)\\
&\approx \frac{\Gamma}{\pi}\frac{a\cdot sin\left(\theta\right)}{h}
\end{aligned}$$

Because we left out some of the r^2terms due to the fact that they are much smaller than h^2terms. Still this equation does not seem to correspond reality. There might be an error in it. However, let's still try to get some result. So the velocity square could be written as:

$$\begin{aligned}
v_\theta^2 &= \left(-2\frac{u}{a^2}sin\left(\theta\right)\left(\frac{1}{h^2}+1\right)+\frac{\Gamma}{\pi}\frac{a\cdot sin\left(\theta\right)}{h}\right)^2\\
&= 4\frac{u^2}{a^4}sin^2\left(\theta\right)\left(\frac{1}{h^2}+1\right)^2-4\frac{u}{a}\frac{\Gamma}{\pi}\left(\frac{1}{h^2}+1\right)sin\left(\theta\right)+\frac{\Gamma^2}{\pi^2}\frac{a\cdot sin^2\left(\theta\right)}{h^2}=\\
&= -4\frac{u}{a^2}\frac{\Gamma}{\pi}\left(\frac{1}{h^2}+1\right)sin\left(\theta\right)
\end{aligned}$$

We are interested only about those terms which do not vanish in integral. So

$$dF_y=-L\cdot a\cdot d\theta\cdot P\left(\theta\right)\cdot sin\left(\theta\right)$$

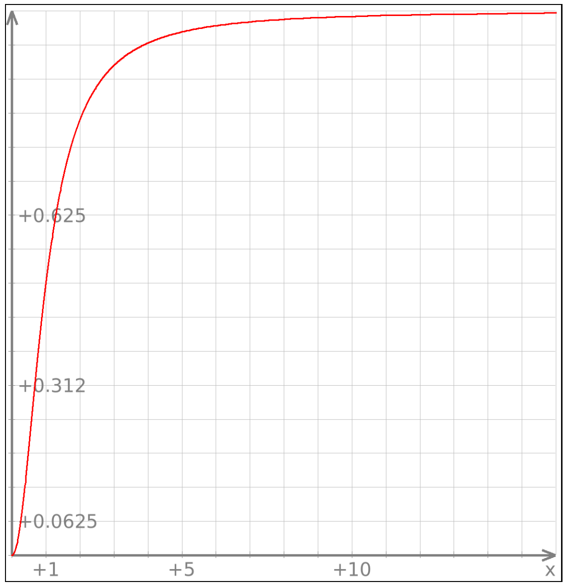

Figure 11: Velocity dependence on proximity of the surface. X axis is the h and Y is the velocity.

and the integral is:

$$\begin{aligned} F_y &= 2L \cdot \rho_0 \frac{u}{a}\frac{\Gamma}{\pi}\left(\frac{1}{h^2}+1\right)\int_0^{2\pi} sin^2(\theta)d\theta \\ &= L \cdot \rho_0 \frac{u}{a}\Gamma\left(\frac{1}{h^2}+1\right) \end{aligned}$$

0.5.4 Support against gravity

We use the same formula as in previous section:

$$-Mg + L \cdot \rho_0 \frac{u}{a}\Gamma\left(\frac{1}{h^2}+1\right) = 0$$

$$u = \frac{Mga}{L\rho_0\Gamma\left(\frac{1}{h^2}+1\right)}$$

The shape of the graph for the velocity is shown in Figure 11. It seems that necessary velocity to keep the plane in the air decreases with approaching to the surface. This theory is supported also by some articles I was able to found and it also seems locigal - ground decreases the flow, making the flow on the top of the plane even more faster relative to the flow at the bootom.