Physics 740: Spring 2007: 04/26/07

P740.16.tex

Viscous Fluid Flow.

1. The equations. We will deal with a viscous fluid in the case that the density is constant. Then the equations for mass conservation and momentum conservation reduce to

$$\nabla \cdot \mathbf{v} = 0, \tag{1}$$

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = -\frac{1}{\rho_0}\nabla P + D_\eta \nabla^2 \mathbf{v}, \tag{2}$$

where $D_\eta = \eta/\rho_0$ is the viscous diffusion constant. See the Table on LL 46 for some numbers.

2. The stress tensor and R_e. Two formal statements about Eqs. (1) and (2).

1. The forces on the RHS of Eq. (2) can be formally arranged to define a stress tensor

$$\frac{\partial v_i}{\partial t} + (\mathbf{v} \cdot \nabla)v_i = -\frac{1}{\rho_0}\frac{\partial P}{\partial x_i} + D_\eta \nabla^2 v_i = \frac{1}{\rho_0}\sum_j \frac{\partial \sigma_{ij}}{\partial x_j}, \tag{3}$$

$$\sigma_{ij} = -P\delta_{ij} + \eta\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right), \tag{4}$$

where the viscosity term has been symmetrized (check to see that this procedure leads to no change in the Navier -Stokes equation, Eq. (2)).

2. The size of the inertial term $(\mathbf{v} \cdot \nabla)v_i$ and the viscous stress term are

   (a) inertial $(\mathbf{v} \cdot \nabla)v_i \sim U^2/L$,

   (b) viscous stress $D_\eta \nabla^2 v_i \sim D_\eta U/L^2$.

   The ratio, defined to be the **Reynolds number**, is

$$R_e = \frac{UL}{D_\eta} = \frac{L^2/D_\eta}{L/U} = \frac{\tau_D}{\tau_U}. \tag{5}$$

   Here τ_D is the time to diffuse the distance L and τ_U is the time to travel that distance at velocity U. When $R_e \ll 1$ diffusion over L is much faster than fluid flow. In the opposite limit, $R_e \gg 1$ the diffusion and viscosity are relatively unmportant.

   http://www.aip.org/pt/jan00/berg.htm. Life at $R_e = 10^{-4}$.

   E. coli, a self-replicating object only a thousandth of a millimeter in size, can swim

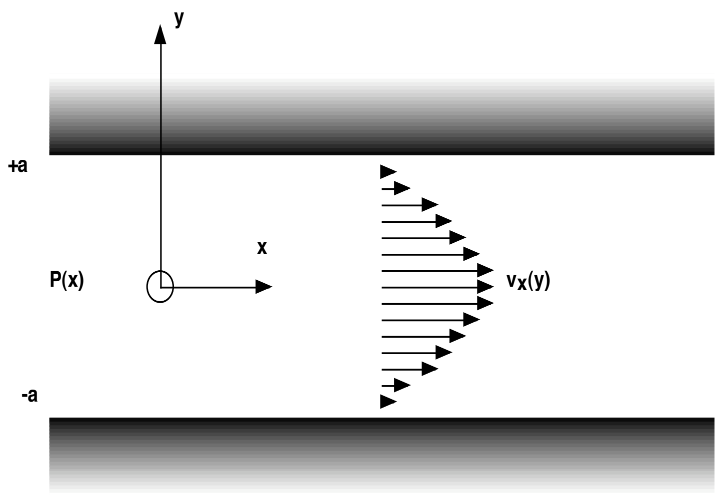


FIG. 1: Poiseuille Flow.

35 diameters a second, taste simple chemicals in its environment, and decide whether life is getting better or worse.

– Howard C. Berg

http://socrates.berkeley.edu/ shaevitz/Movies/Spiroplasma%20Kinking.mov

3. Viscosity at work; two examples.

3a. Poiseuille Flow

Consider a fluid between two parallel plates, separated by $2a$, that flows in response to a steady pressure gradient. See Fig. 1. Since $\mathbf{v} = (v_x(y), 0, 0)$ the $(\mathbf{v} \cdot \nabla)v_x$ term in Eq. (2) is zero and we have

$$-\frac{1}{\rho_0}\frac{\partial P}{\partial x} + D_\eta \frac{\partial^2 v_x}{\partial y^2} = 0. \tag{6}$$

For a non slip boundary condition at $y = \pm a$ we have

$$\rho_0 v_x(y) = \frac{1}{2}\frac{a^2}{D_\eta}\left|\frac{\partial P}{\partial x}\right|\left(1 - \frac{y^2}{a^2}\right) = \frac{1}{2}\tau_\eta \left|\frac{\partial P}{\partial x}\right|\left(1 - \frac{y^2}{a^2}\right), \tag{7}$$

a parabolic velocity profile for which the maximum velocity, at $y = 0$, is proportional to $\tau_\eta = a^2/D_\eta$. This fluid flow, called Poiseuille flow, is controlled by the rate at which the viscosity carries momentum, delivered to the fluid by the pressure gradient, to the walls of

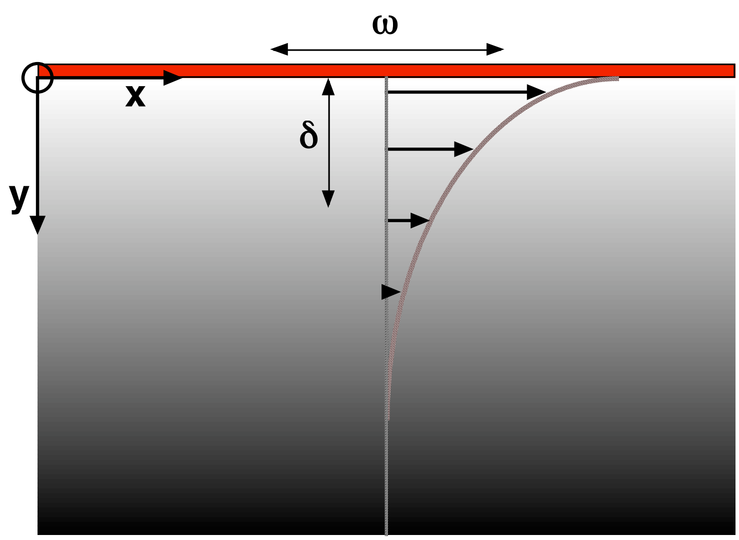


FIG. 2: Viscous penetration depth.

the flow space where it is lost. Hence the involvement of τ_η.

1. The total flow of fluid $Q = \int \rho_0 v_x(y)\ dy \propto a^3$.

2. For a circular pipe expect $Q \propto a^4$, LL (17.10).

3. For a current in a conductor $I = V/R$ and $R = \rho L/A$ (here ρ is the resisitivity, L the length of the conductor and $A \propto a^2$ the cross sectional area). Thus $I \propto a^2$

3b. Viscous penetration depth.

See Fig. 2. Suppose the upper surface of a fluid is in contact with a solid surface that is oscillated at frequency ω. The fluid "sticks" to the surface so the upper edge of the fluid moves with velocity $v_x(y = 0) = Aexp\ (-i\omega t)$. The equation of motion for the fluid is

$$\frac{\partial v_x}{\partial t} = D_\eta \frac{\partial^2 v_x}{\partial y^2}, \tag{8}$$

the x-component of Eq. (2), P does not depend on x and the inertial term is zero because v_x depends only on y. For $v_x(y) = U(y)exp\ (-i\omega t)$ find

$$\frac{\partial^2 U}{\partial y^2} = (-i\omega/D_\eta)U = -\kappa^2 U, \tag{9}$$

and

$$U(y) = A\ exp\ (i\kappa y) = A\ exp\ (ik_1 y)\ exp\ (-k_1 y) = A\ exp\ (ik_1 y)\ exp\ (-y/\delta_\eta) \tag{10}$$

where $k_1^2 = \omega/2D_\eta$ and δ_η is the viscous penetration depth. The disturbance at the surface at frequency ω penetrates into the interior of the liquid to the approximate depth to which momentum can diffuse on time scale $T_\omega \approx \omega^{-1}$, $D_\eta T_\omega \approx \delta_\eta^2$.

The physics here is similar to that associated with the skin depth in E and M. In that case an electric field on the surface of a conductor attempts to move the electron gas within the conductor at frequency ω. It succeeds to depth

$$\delta_\sigma^2 \approx \frac{c^2/\sigma}{\omega} \approx D_\sigma T_\omega, \tag{11}$$

where $c^2/\sigma = D_\sigma$, σ is the conductivity. Jackson, Sec. 7.7.

4. Low Reynolds number flow, the Stokes problem.

The problem is that of a sphere, radius R, moving through a viscous fluid with velocity $\mathbf{u} = (0, 0, u)$. The problem solved is that of a sphere at rest with a fluid, having velocity $\mathbf{u}$ at large distance, moving past the sphere. This is a steady flow problem; the equations to be solved are

$$\nabla \cdot \mathbf{v} = 0, \quad -\nabla P + \eta \nabla^2 \mathbf{v} = 0. \tag{12}$$

This is an algebra intensive problem. Begin by removing the pressure from the problem (we can come back and *learn it* at the end. [Here the pressure is an output, not an input.] Take **curl** of the second of Eq. (12) and use $\nabla \cdot \mathbf{u} = 0$ to find the two equations of interest

$$\nabla \cdot (\mathbf{v} - \mathbf{u}) = 0, \quad \nabla^2\ \nabla \times \mathbf{v} = 0. \tag{13}$$

From first of these suggests writing $\mathbf{v} - \mathbf{u}$ as the **curl** of a vector field so that $\nabla \cdot$ will be automatically satisfied. Then, it is from the second that the conditions on this potential are established.

$$\mathbf{v} - \mathbf{u} = \nabla \times \mathbf{A}. \tag{14}$$

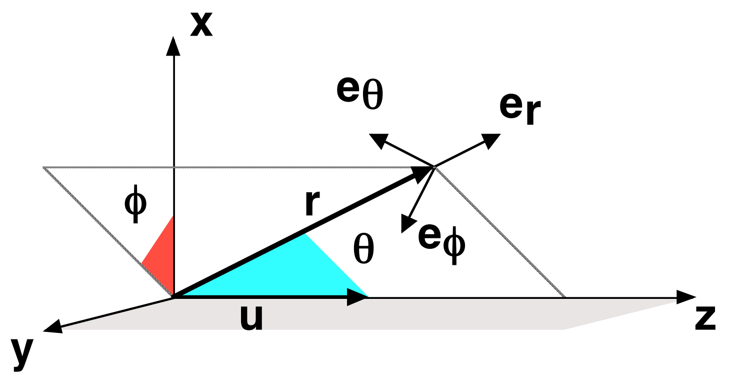


FIG. 3: Coordinate system.

The velocity field has **no** components in the $\mathbf{e}_\phi$ direction, by symmetry, Fig. 3. Look in Griffiths or Jackson; the vector potential points in that direction so that there are non-zero e_r and e_θ components. So the vector potential can be written in the form

$$A = g(r)\mathbf{e}_r \times \mathbf{u} = g(r)\ \mathbf{e}_r \times u\mathbf{e}_z = \nabla f(r) \times \mathbf{u} = \nabla \times (f(r)\mathbf{u}), \tag{15}$$

since $\mathbf{e}_r \times \mathbf{e}_z = \mathbf{e}_\phi$. We need to find $f(r)$.

From Eqs. (14) and (13)

$$\mathbf{v} = \mathbf{u} + \nabla \times \mathbf{A} = \mathbf{u} + \nabla \times \nabla \times (f(r)\mathbf{u}), \tag{16}$$

$$\nabla \times \mathbf{v} = \nabla \times \nabla \times \nabla \times (f(r)\mathbf{u}), \tag{17}$$

$$= [\nabla(\nabla\cdot) - \nabla^2]\ \nabla \times (f(r)\mathbf{u}), \tag{18}$$

$$= -\nabla^2\ \nabla \times (f(r)\mathbf{u}), \tag{19}$$

$$\nabla^2\ \nabla \times \mathbf{v} = -\nabla^2\ \nabla^2\ \nabla \times (f(r)\mathbf{u}) = 0, \tag{20}$$

$$= -\nabla^2\ \nabla^2\ \nabla(f(r) \times \mathbf{u}) = 0, \tag{21}$$

$$\nabla\ \nabla^2\ \nabla^2\ f(r) = 0. \tag{22}$$

This evolution uses such things as $\nabla \cdot \nabla \times\ = 0$, the expansion for **curl curl**, etc. Since f only depends on r $\nabla = \partial/\partial r$ etc. There are 5 derivatives of f in the final formula. We want to integrate this equation. The first integral is $\nabla^2\ \nabla^2 f = constant = 0$, where the choice $constant = 0$ comes from hindsight. To go on

$$
\begin{aligned}
\nabla^2\ \nabla^2 f &= \nabla^2 h = 0, \quad \nabla^2 f = h, && (23)\\
\nabla^2 h &= \frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial h}{\partial r} = 0, \quad h = \frac{2a}{r}, && (24)\\
\nabla^2 f &= h = \frac{2a}{r}, && (25)\\
f &= ar + \frac{b}{r}, && (26)
\end{aligned}
$$

where a and b are constants to be determined. Use $f(r)$ in Eq. (16) and find

$$
\mathbf{v} = \mathbf{u} - \frac{a}{r}\left[\mathbf{u} + \mathbf{e}_r(\mathbf{u}\cdot\mathbf{e}_r)\right] - \frac{b}{r^3}\left[\mathbf{u} - 3\mathbf{e}_r(\mathbf{u}\cdot\mathbf{e}_r)\right]. \quad (27)
$$

To find a and b ask that $\mathbf{v} = 0$ on the surface of the sphere. As $\mathbf{v}$ in Eq, (27) is built up from two independent vectors, $\mathbf{u}$ and $\mathbf{e}_r$, the amplitude of each must vanish at $r = R$. Thus

$$
a = \frac{3}{4}R, \quad b = \frac{1}{4}R^3 \quad (28)
$$

and

$$
\frac{\mathbf{v}}{u} = \left[1 - \frac{3R}{2r} + \frac{R^3}{r^3}\right]\ cos\theta\ \mathbf{e}_r - \left[1 - \frac{3R}{4r} - \frac{R^3}{4r^3}\right]\ sin\theta\ \mathbf{e}_\theta. \quad (29)
$$

Before looking at this result let's go on and calculate a few other things of eventual interest. We find the pressure associated with the flow by going back to Eq. (12). Again some algebra

$$
\begin{aligned}
\nabla P &= \eta\nabla^2\mathbf{v} = \eta\nabla^2\ \nabla\times\nabla\times(f\mathbf{u}), && (30)\\
\nabla P &= \eta\nabla^2\ [\nabla(\nabla\cdot) - \nabla^2](f\mathbf{u}), && (31)\\
\nabla P &= \eta\nabla^2\ \nabla(\nabla\cdot)(f\mathbf{u}) = \eta\nabla\ \nabla^2\ (\nabla\cdot f\mathbf{u}), && (32)\\
P &= \eta\nabla^2\ (\nabla\cdot f\mathbf{u}), && (33)\\
P &= \eta\nabla^2\ (\mathbf{u}\cdot\nabla f) = \eta(\mathbf{u}\cdot\nabla)\ \nabla^2\ f = \eta(\mathbf{u}\cdot\nabla)\ \frac{3}{2}\frac{R}{r}. && (34)\\
& && (35)
\end{aligned}
$$

And finally

$$
P = -\frac{3\eta u}{2R}\ (\mathbf{e}_z\cdot\mathbf{e}_r)\ \frac{R^2}{r^2}. \quad (36)
$$

Armed with this expression and the equation for $\mathbf{v}$ one can learn the net force on the sphere. Again there is much detail See LL (20.13) et. seq. Except for numbers the answer must scale as P times the area that P works on , i.e., πR^2. The force is opposite to the direction the sphere is moving

$$\mathbf{F} \propto R^2 \times P(r = R) \approx \eta R u = -6\pi\eta R u \mathbf{e}_u. \tag{37}$$

Summary

$$\frac{\mathbf{v}}{u} = \left[1 - \frac{3R}{2r} + \frac{R^3}{r^3}\right] cos\theta \; \mathbf{e}_r - \left[1 - \frac{3R}{4r} - \frac{R^3}{4r^3}\right] sin\theta \; \mathbf{e}_\theta, \tag{38}$$

$$P = -\frac{3\eta u}{2R} \left(\mathbf{e}_z \cdot \mathbf{e}_r\right) \frac{R^2}{r^2}, \tag{39}$$

$$\mathbf{F} = -6\pi\eta R u \mathbf{e}_u. \tag{40}$$

5. Terminal velocity and such.

A sphere of radius R, density ρ_M, in a viscous liquid and in the presence of gravity obeys the equation of motion

$$M\dot{u} = -6\pi\eta R u \mathbf{e}_u - Mg, \tag{41}$$

where u is the velocity of the sphere and M its mass. Re-arrange this to read

$$\dot{u} = -\frac{1}{\tau_S} u - g, \tag{42}$$

where

$$\frac{1}{\tau_S} = \frac{6\pi\eta R}{M}. \tag{43}$$

Solution to this equation is

$$u(t) = u(0) e^{-\frac{t}{\tau_S}} + g\tau_S (1 - e^{-\frac{t}{\tau_S}}). \tag{44}$$

1. The initial velocity decays away on time scale τ_S.

2. The terminal velocity, $u_T = g\tau_S$ is established on the same time scale.

3. You can read the terminal velocity from Eq. (42) with $\dot{u} = 0$. It is that velocity for which the Stokes drag balances the driving force.

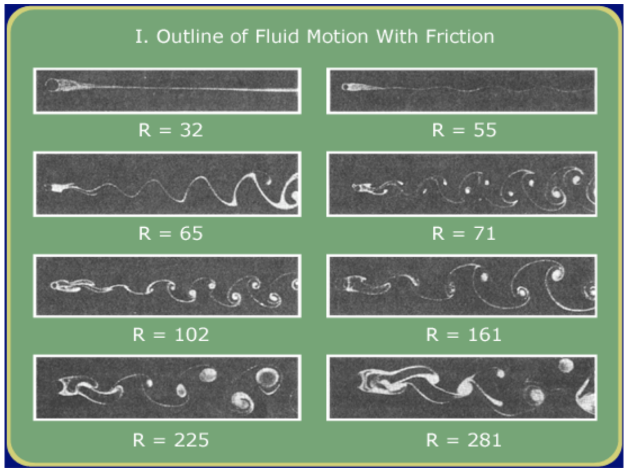
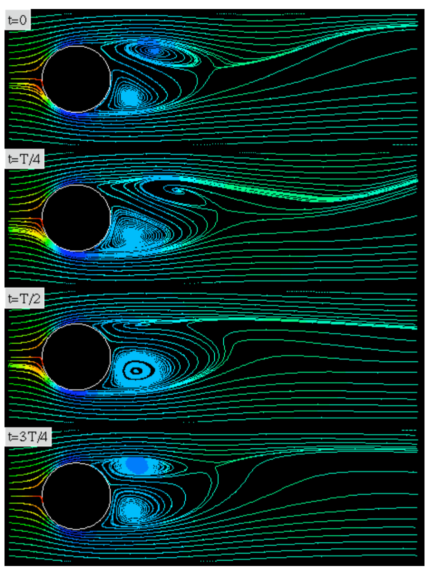


FIG. 4:



Re = 270

FIG. 5:

6. High Reynolds number.