P740.HW3.sol.tex

1. The atmosphere has scale height $z_T = k_B T/(mg)$. While careful calculation is possible the essential content (only some numbers will be different,and not by much) is that the size of the atmosphere is

$$V_A = 4\pi R^2 z_T. \tag{1}$$

The density of gas in the atmosphere is approximately $\rho(0) = mN/V$ where $N/V = P(0)/(k_B T)$. Thus

$$M_A = \rho(0)V_A = 4\pi R^2 \frac{P(0)}{g} \approx 5 \times 10^{21}\ grams. \tag{2}$$

The mass of the earth is $M_E \approx 6 \times 10^{27}\ grams$. Annual emission of carbon into atmosphere by US is $2 \times 10^{15}\ grams$.

2. With the equation of state of the material the equation of hydrostatic equilibrium is

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \rho}{\partial r}\right) = \rho^{''} + \frac{2}{r}\rho^{'} = -\kappa^2 \rho, \tag{3}$$

where $\kappa^2 = 4\pi G \rho_0^2/P_0$. A possible way to solve this equation is with the substitution $\rho = Q/r$ (often used in QM for D=3 problems, but not always). Find

$$Q^{''} = -\kappa^2 Q. \tag{4}$$

This equation has solutions in terms of $\sin \kappa r$ and $\cos \kappa r$. At the star radius, $r = R$, we have $P = 0$ and from the EOS $\rho = 0$. So choose

$$\rho = A \frac{\sin \kappa (R - r)}{r}. \tag{5}$$

Note: that if $\kappa R > \pi$ it is possible that the star will have negative density. Is that OK?

Fix A by requiring that the total mass of the star be M. Thus

$$M = A\ 4\pi \int_0^R r \sin \kappa (R - r)\ dr. \tag{6}$$

It is useful to scale r by R and ρ by $\overline{\rho} = M/V_R$ ($V_R = 4\pi R^3/3$) here, possibly earlier or later. For example

$$A = \frac{M}{V_R} 3R \frac{1}{\int_0^1 x \sin \alpha (1 - x)\ dx}, \tag{7}$$

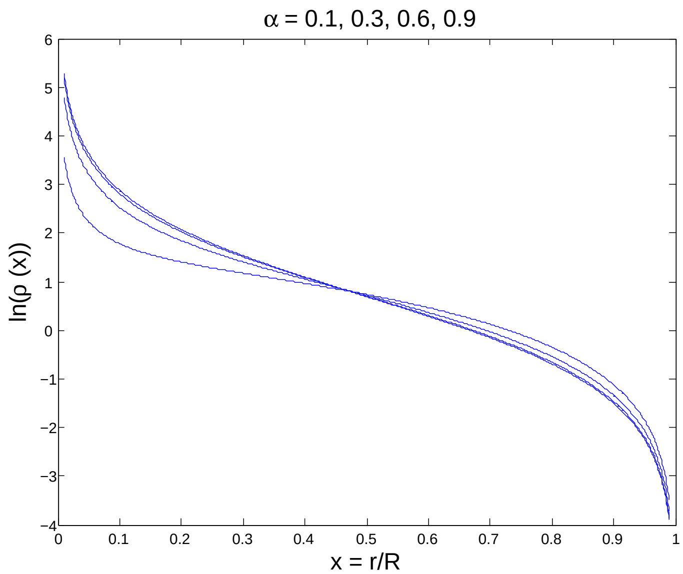

FIG. 1: $\ln(\rho(x))$ as a function of x for 4 values of α

where $\alpha = \kappa R$. Find A and rearrange

$$\rho^* = \frac{\rho}{\bar{\rho}} = \frac{\alpha}{3(1 - S_\alpha)} \frac{\sin \alpha(1 - x)}{x}. \tag{8}$$

where $S_\alpha = \sin \alpha / \alpha$. See Fig. 1.

3. Start with Eqs. (33)=(35) in linear form, equilibrium at $(\rho_0, P_0, \theta_0, \mathbf{u}_0 = 0)$ and first order terms $(\delta\rho, \delta P, \delta\theta, \mathbf{u})$:

$$\delta\dot{\rho} + \rho_0 \nabla \cdot \mathbf{u} \ = \ 0, \tag{9}$$

$$\dot{\mathbf{u}} = -\frac{1}{\rho_0}\nabla\delta P + \frac{D_\eta}{3}\nabla(\nabla\cdot\mathbf{u}) + \nabla^2\mathbf{u}, \tag{10}$$

$$\delta\dot{\theta} = -\frac{\theta_0}{C_V}\nabla\cdot\mathbf{u} + D_\kappa\nabla^2\delta\theta, \tag{11}$$

$$\delta P = \Lambda_\rho\delta\rho + \Lambda_\theta\delta\theta, \tag{12}$$

where $\Lambda_x = \partial P/\partial x$. Use

$$\delta\rho = A\ exp\ i(\mathbf{k}\cdot\mathbf{x} - \omega t), \tag{13}$$

$$\delta\theta = B\ exp\ i(\mathbf{k}\cdot\mathbf{x} - \omega t), \tag{14}$$

$$\delta P = H\ exp\ i(\mathbf{k}\cdot\mathbf{x} - \omega t), \tag{15}$$

$$\mathbf{u} = \mathbf{C}\ exp\ i(\mathbf{k}\cdot\mathbf{x} - \omega t). \tag{16}$$

$$\tag{17}$$

and find

$$-i\omega A + \rho_0 i\mathbf{k}\cdot\mathbf{C} = 0, \tag{18}$$

$$-i\omega\mathbf{C} = -i\mathbf{k}\frac{H}{\rho_0} - \frac{D_\eta}{3}\mathbf{k}(\mathbf{k}\cdot\mathbf{C}) - k^2 D_\eta\mathbf{C}, \tag{19}$$

$$-i\omega B = -\frac{\theta_0}{C_V}i\mathbf{k}\cdot\mathbf{C} - D_\kappa k^2 B, \tag{20}$$

$$H = \Lambda_\rho A + \Lambda_\theta B. \tag{21}$$

Solve Eqs. (10) and (12) for use in Eqs. (13) and Eq. (11)

$$\frac{H}{\rho_0} = \left(\Lambda_\rho + \frac{\Lambda_\theta\theta_0}{\rho_0 C_V}\frac{\omega}{\omega + ik^2 D_\kappa}\right)\frac{\mathbf{k}\cdot\mathbf{C}}{\omega} = c^2(\omega)\frac{\mathbf{k}\cdot\mathbf{C}}{\omega}, \tag{22}$$

$$i\omega\mathbf{C} = i\mathbf{k}\ c^2(\omega)\ \frac{\mathbf{k}\cdot\mathbf{C}}{\omega} + \frac{D_\eta}{3}\mathbf{k}(\mathbf{k}\cdot\mathbf{C}) + k^2 D_\eta\mathbf{C}. \tag{23}$$

Dot this equation with $\mathbf{k}$, remove $\mathbf{k}\cdot\mathbf{C}$ and re-arrange

$$i\omega = ik^2\ c^2(\omega)\ \frac{1}{\omega} + \frac{D_\eta}{3}k^2 + k^2 D_\eta. \tag{24}$$

$$\omega^2 = k^2\ c^2(\omega) - i\frac{4D_\eta}{3}k^2\omega. \tag{25}$$

We are not done yet because $c^2(\omega)$ is complicated and complex. Sort it into a real and imaginary parts

$$c^2(\omega) = c_0^2(\omega) - ic_R^2\ \frac{\frac{k^2 D_\kappa}{\omega}}{1 + \left(\frac{k^2 D_\kappa}{\omega}\right)^2} \tag{26}$$

where

$$c_0^2(\omega) = c_T^2 + c_R^2 \, \frac{1}{1 + \left(\frac{k^2 D_\kappa}{\omega}\right)^2} \tag{27}$$

where $c_T^2 = (\partial p/\partial \rho)_T$ is the isothermal sound speed and

$$c_R^2 = \frac{\Lambda_\theta \theta_0}{\rho_0 C_V}. \tag{28}$$

Finally

$$\omega^2 = k^2 \; c_0^2(\omega) - i\frac{4}{3}k^2 \; D_\eta \; \omega - ik^2 c_R^2 \; \frac{\frac{k^2 D_\kappa}{\omega}}{1 + \left(\frac{k^2 D_\kappa}{\omega}\right)^2} = k^2 \; c_0^2(\omega) - iR_\eta - iR_\kappa. \tag{29}$$

Note

1. $c_0^2(\omega)$. As $\omega \to 0$, $c_0^2 \to c_T^2$, the isothermal sound speed. As $\omega \to +\infty$, $c_0^2 \to c_T^2 + c_R^2$, which must be the adiabatic sound speed. The thermal diffusivity and (k, ω) control the trasition from isothermal to adiabatic.

2. Damping. There are two damping terms, one involving the viscosity and a second involving the thermal diffusivity. The damping when both terms are present is the sum of two independent damping terms. They do not interfere with one another. The damping processes are addative. This is like resistors in series. Resistors in parallel interfere with one another. For two resistors in parallel the current through one can only be learned when you know the size of the second.