Physics 740: Spring 2006: 02/22/07

P740.8.tex

Vortices II. Examples. The basic equations involved in determining the motion of a system of vortices are Eqs. (7)-(10) on Note P740.7.tex. With the *sign* convention + = arrow out of page and − = arrow into page the motion of the vortices in Figs 1 and 2 can be understood qualitatively. Equations (9) and (10) can be derived from the energy

$$\mathcal{E} = N\epsilon_c - \frac{1}{4\pi}\sum_{i\neq j}\kappa_i\kappa_j\; ln(r_{ij}/2a), \tag{1}$$

where $r_{ij} \geq 2a$ is the separation between the core centers of vortices having circulation κ_i and κ_j and a is the core radius. Notice the *minus* sign in the equation. Two vortices of the same sign of circulation have minimum energy at $r \rightarrow \infty$ (like signs repel). Two vortices of different sign of circulation have minimum energy at $r \rightarrow 2a$ (opposite signs attract?). Except for a few details the energy $\mathcal{E}$ is of the same form as that of a D=2 gas of charges. The equations of motion, Eqs. (9) and (10) can be found from

$$\kappa_i\dot{x}_i = \frac{\partial\mathcal{E}}{\partial y_i}, \quad \kappa_i\dot{x}_i = -\frac{\partial\mathcal{E}}{\partial x_i}. \tag{2}$$

1. Figure 1. Four examples of the motion of pairs of vortices. How does the motion seen compare with that of the analogous charge system? If it differs markedly? Why?

2. Figure 2. Two vortices that circle a common center. The smooth curve (in each panel seen at 4 moments of time during the vortex motion pictured) is a *path in the fluid* about which you might have chosen to calculate the circulation. This path moves in time sort of winding up as seen in the lower right panel. This is an illustration of the sort of motion a path in the fluid undergoes. The Kelvin Theorem say the circulation around this path is a constant even as the path distorts.

3. Figure 3. Two vortices leapfrog one another.

4. Figure 4. The collision of two vortices (smoke rings) initially separated by 30 units and headed toward one another. They both have about the same speed.

5. Figure 5. The collision of two vortices (smoke rings) initially separated by 30 units and headed toward one another. The vortex on the right is smaller and faster.

6. Figure 6. A smaller, faster vortex passes through a larger, slower vortex and keeps on going. There must be an upper limit to leapfroging.

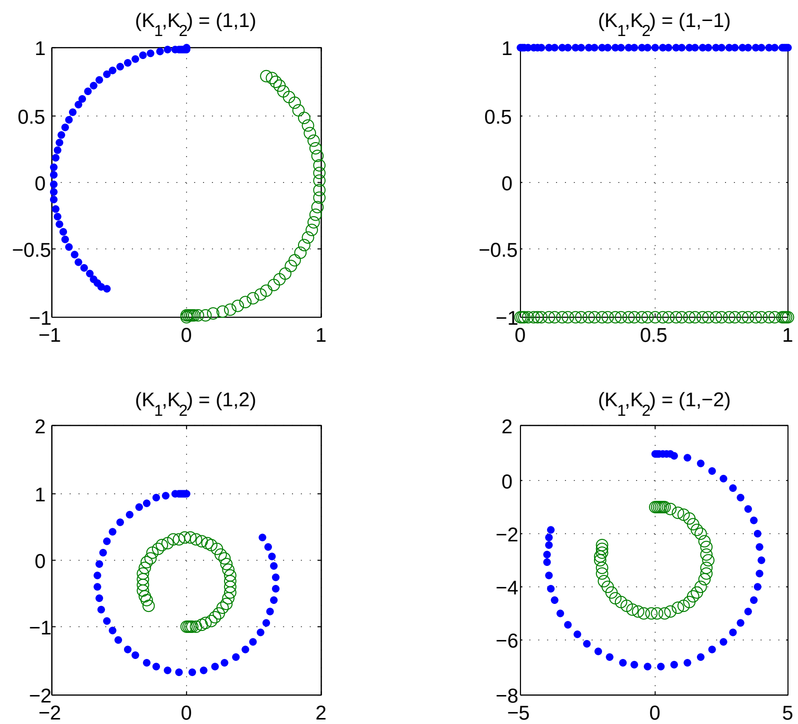

FIG. 1: Motion of pairs. The vortex κ_1 (κ_2) is initially at $(0,1)$ ($(0,-1)$).

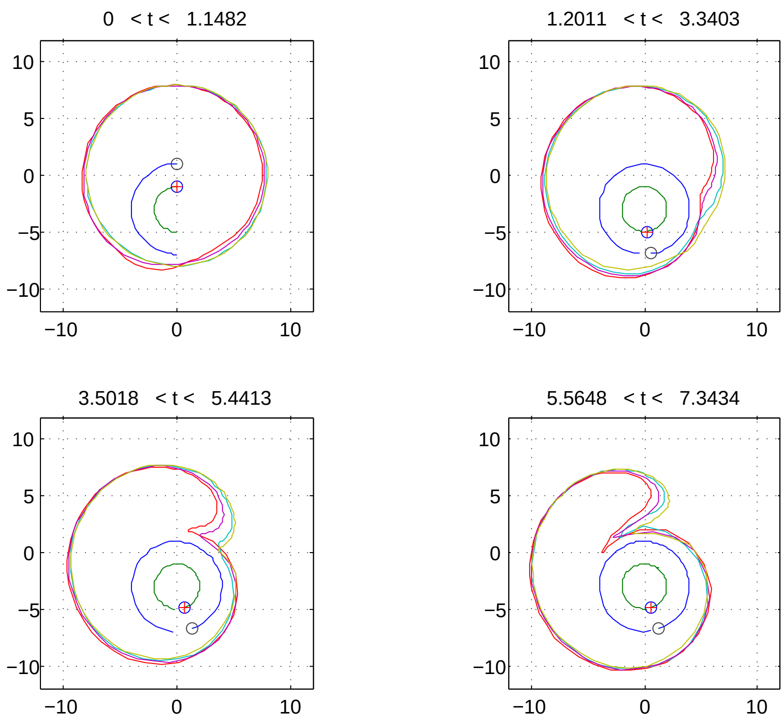

FIG. 2: Two vortices and a loop in the fluid. The loop at 4 moments of time during each time segment is shown as a solid line. Over 3 and 1/2 periods of the vortex motion the loop begins to wind up. The vortices remain inside.

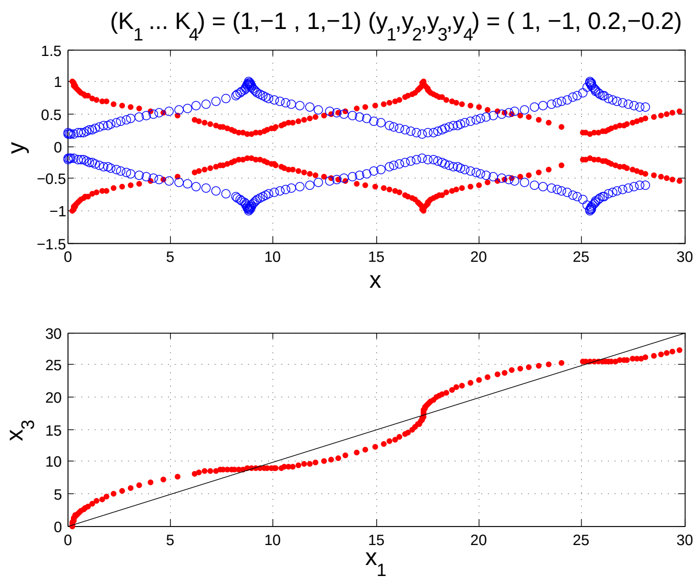

FIG. 3: Two vortex pairs (two smoke rings) can leapfrog one another. The smoke rings have the same circulation and sense of motion. The cores (1,2) of the initially larger vortex are at $(x_{1,2}, y_{1,2}) = (\pm 1, 0.5)$ and the cores (3,4) of the initially smaller vortex are at $(x_{3,4}, y_{3,4}) = (\pm 0.2, 0.0)$. (a) Upper panel. The initially faster vortex slows and the initially slower vortex speeds up. And then they exchange. (b) Lower panel. The x-position of the initially smaller vortex as a function of the x-position of the initially slower vortex, i.e., x_3 as a function of x_1.

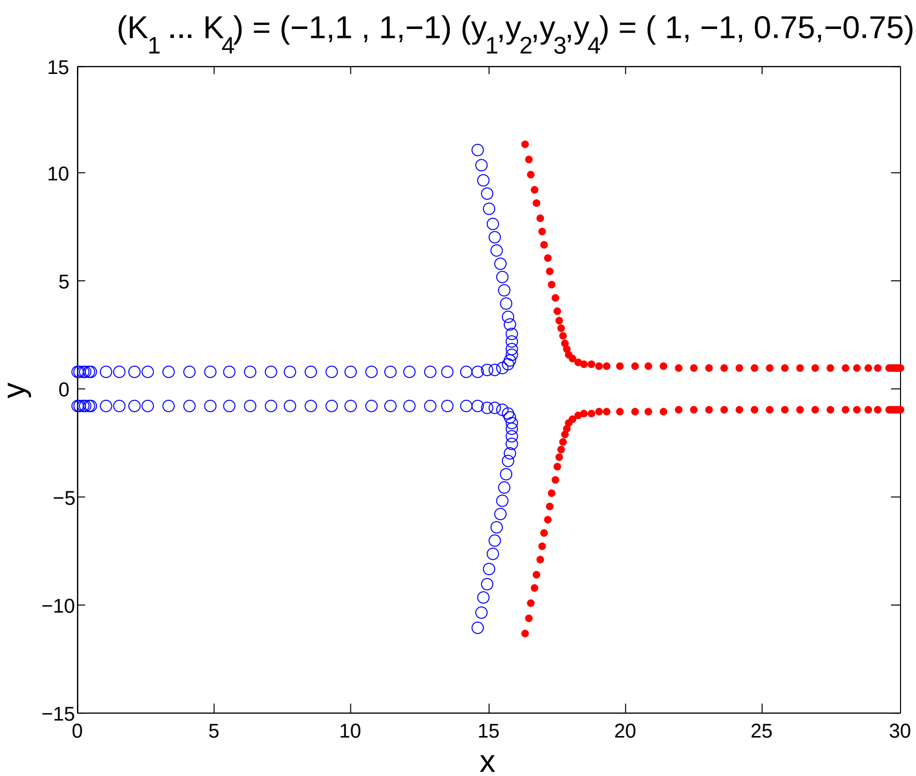

FIG. 4: Collision 1. Two smoke rings with the opposite sense of rotation can approach one another and scatter. See Fig. 5.

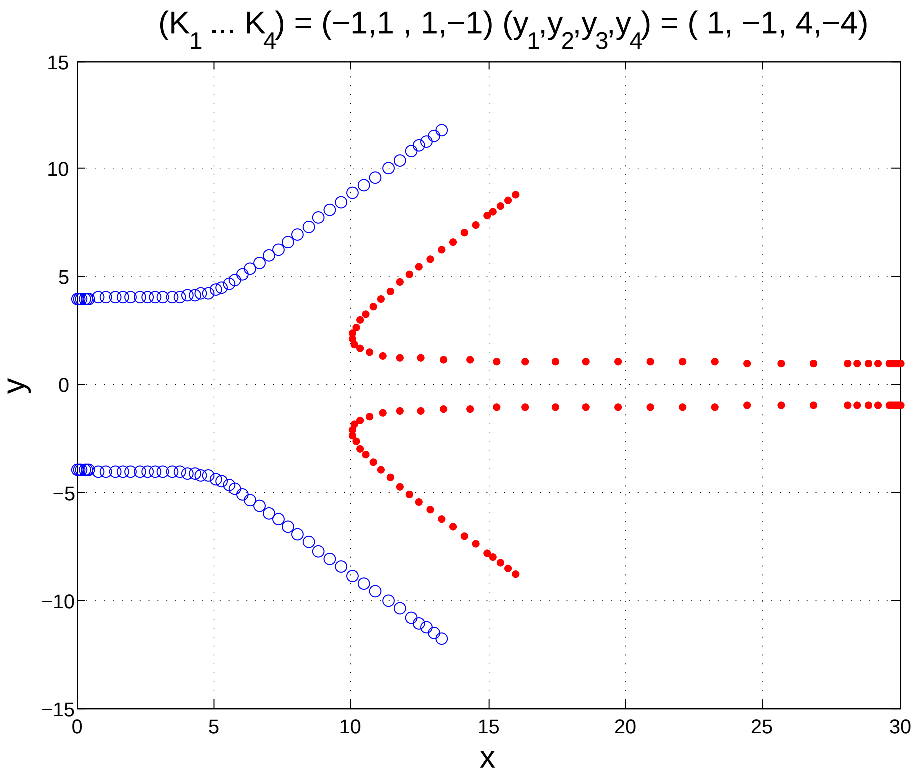

FIG. 5: Collision 2. Two smoke rings with the opposite sense of rotation can approach one another and scatter. See Fig. 4.

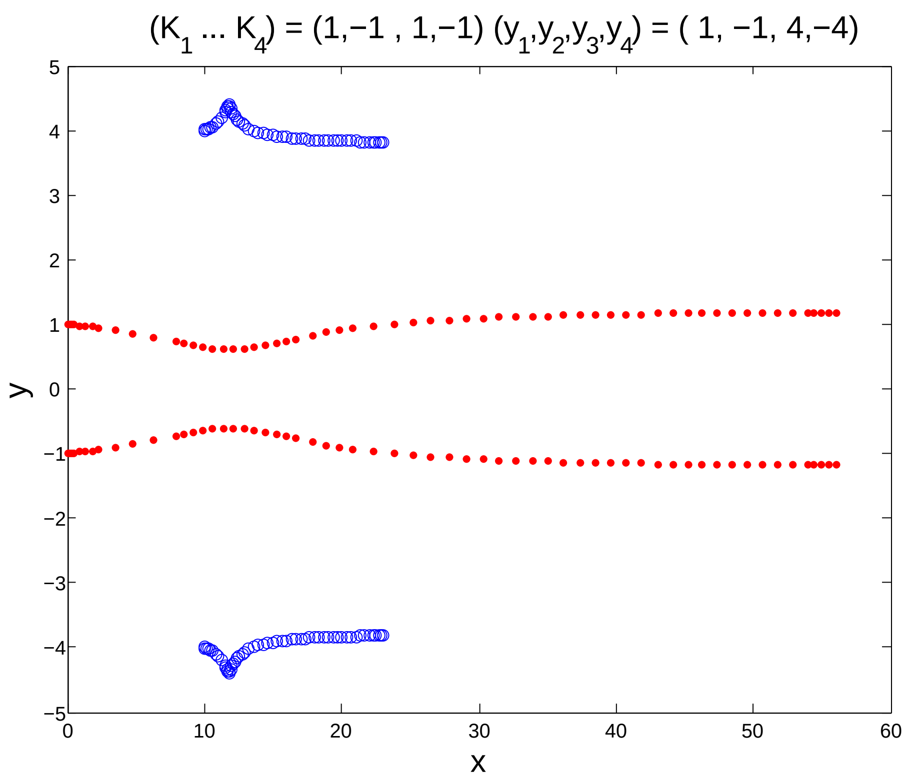

FIG. 6: Collision 3. Two smoke rings with the same sense of rotation need not leapfrog. Particularly if one is very fast. See Fig. 3.