Physics 740: Spring 2006: 02/26/07

P740.9.tex

Ideal Fluids in D=2. For fluids described by the Euler equation, the continuity equation and an equation for energy conservation, there are no transport processes, $\eta = 0$, $\kappa_T = 0$ and the entropy of each piece of fluid is fixed. Note **P740.6.tex**, Eqs. (1)-(3). Such a transportless fluid (sometimes called inviscid) may also have constant density, $\rho = \rho_0$.

[When this is the case you don't have to worry about an equation of state for now there are as many equations a variables and the pressure, P, whatever it happens to do, does not vary with ρ since ρ is constant.]

1. Ideal Fluid. For an ideal fluid we have $\eta = 0$, $\kappa_T = 0$, $\rho = \rho_0$ and the entropy of each piece of fluid is fixed. Because ρ is constant the continuity equation reduces to

$$\nabla \cdot \mathbf{v} = 0. \tag{1}$$

On our way to the Bernoulli equation we used the Euler equation in the form

$$\frac{\partial \mathbf{v}}{\partial t} - \mathbf{v} \times (\nabla \times \mathbf{v}) = -\nabla \left(h + \frac{1}{2} v^2 \right). \tag{2}$$

If we take the **curl** of this equation we have

$$\frac{\partial \left(\nabla \times \mathbf{v}\right)}{\partial t} - \nabla \times \mathbf{v} \times (\nabla \times \mathbf{v}) = 0. \tag{3}$$

This equation, equivalent to the Euler equation, is solved by $\nabla \times \mathbf{v} = 0$. Thus the continuity equation and the Euler equation reduce to finding $\mathbf{v}$ such that

$$\nabla \cdot \mathbf{v} = 0, \quad \nabla \times \mathbf{v} = 0. \tag{4}$$

We must complement these equations with information on how $\mathbf{v}$ behaves asymptotically in space and how it behaves near physical surfaces. Subtlety near surfaces ocurs when the viscosity is present. Without η we simply don't let the fluid pass through a surface. We have boundary condition $\mathbf{v} \cdot \mathbf{n}$=0, where $\mathbf{n}$ is the pointwise normal to a surface.
The ideal fluid problem is set by

$$\nabla \cdot \mathbf{v} = 0, \quad \nabla \times \mathbf{v} = 0, \quad \mathbf{n} \cdot \mathbf{v} \; on \; \Sigma. \tag{5}$$

2. The velocity potential. The first of Eq. (4) is solved trivially if we take

$$\mathbf{v} = \nabla\phi, \quad \nabla\cdot\mathbf{v} = \nabla^2\phi = 0, \tag{6}$$

where ϕ is the velocity potential. To find $\mathbf{v}$ solve $\nabla^2\phi = 0$ subject to appropriate boundary conditions. [As a Math-Physics problem this should remind you of electrostatics.]
When doing two dimensional problems there is a practical advantage afforded by introducing a second scalar field, ψ called the stream function. **From here forward we are in D=2.**

3. The stream function. Define a scalar function ψ such that

$$v_x = u = \frac{\partial\psi}{\partial y}, \quad v_y = v = -\frac{\partial\psi}{\partial x}, \tag{7}$$

where for simplicity we will use $\mathbf{v} = (u, v, w) = (u, v, 0)$, a more or less standard notation, to denote the components of $\mathbf{v}$. Right away we have

$$\nabla\cdot\mathbf{v} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \frac{\partial^2\psi}{\partial x\partial y} - \frac{\partial^2\psi}{\partial y\partial x} = 0 \tag{8}$$

from Eq. (6). With $\mathbf{v}$ given by Eq. (6) we have

$$\nabla\times\mathbf{v} = \mathbf{e}_z\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = -\mathbf{e}_z\left(\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2}\right), \tag{9}$$

or $\nabla\times\mathbf{v} =$ for $\nabla^2\psi = 0$.

4. A complex potential. Take ϕ and ψ to be the two components of an analytic function in the complex plane, i.e.,

$$F(z) = \phi(z) + i\psi(z). \tag{10}$$

From Cauchy-Riemann

$$\frac{\partial\phi}{\partial x} = \frac{\partial\psi}{\partial y} = u, \tag{11}$$

$$\frac{\partial\phi}{\partial y} = -\frac{\partial\psi}{\partial x} = v, \tag{12}$$

with $\nabla^2\phi = \nabla^2\psi = 0$.

Litany. Any analytic function in the complex plane describes a possible **ideal fluid** *flow problem.*

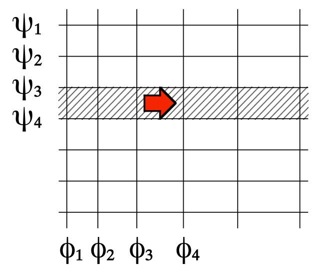

FIG. 1: Fluid flow is in the direction of $\nabla\phi$ and parallel to lines of constant ψ. Lines of constant ψ are orthogonal to lines of constant ϕ.

The analytic function will have two parts, ϕ, the velocity potential, from which you learn $\mathbf{v}$ and ψ, the stream function, from which you learn ... ?

5. About the stream function.

1. Suppose you are on a line of constant ψ, say $\psi = 11$. Then

$$d\psi = \frac{\partial\psi}{\partial x}dx + \frac{\partial\psi}{\partial y}dy = -vdx + udy = 0 \tag{13}$$

   or along a contour of constant ψ

$$\frac{dx}{u} = \frac{dy}{v}. \tag{14}$$

   That is, a contour of constant ψ is a streamline of the fluid flow.

2. The fluid flow between two streamlines is given by the difference between the numerical values of ψ of the streamlines. For a path between two streamlines write

$$\int_2^1 d\mathbf{s}\cdot\mathbf{v} = \int_2^1 (dy, -dx)\cdot(u, v) = \int_2^1 d\psi = \psi_1 - \psi_2. \tag{15}$$

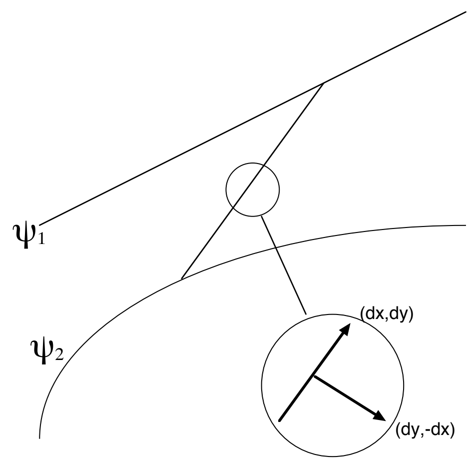

FIG. 2: The fluid flow between two streamlines is given by the difference between the numerical values of ψ of the streamlines.