HW 3, Fluid Dynamics

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0.1 Problem #1

Lets find out how to find the mass of the atmosphere. We know the relations $P(z) = P(0)exp(\frac{-z}{z_T})$ and $z_t = \frac{k_B T}{mg}$, where z denotes distance from the surface of the Earth. Using the ideal gas law, we can write the following relations:

$$PV = Nk_BT \rightarrow P = \frac{mN}{V}\frac{k_BT}{m} \rightarrow P = \rho\frac{k_BT}{m},$$

where m is the average mass of an air molecule. We also should consider limits. In my calculations I assume that thickness of the atmosphere is about 100km and I use $R_{atm} = 100km$ as upper limit in the integral I'm about to calculate. Knowing that, it could be written for the mass of the atmosphere:

$$\begin{aligned} M_a &= \int_0^{R_{atm}} 4\pi(R+z)^2\rho(z)dz \\ &= \int_0^{R_{atm}} 4\pi R^2 P(0)exp\left(-\frac{z}{z_T}\right)\frac{m}{k_BT}dz \\ &= \frac{4\pi M_{air}P(0)R^2}{k_BN_AT}\int_0^{R_{atm}} exp\left(-\frac{z}{z_T}\right)dz \\ &= \frac{4\pi M_{air}P(0)R^2}{RT}\left[-\frac{k_BT}{mg}\left(exp\left(-\frac{R_{atm}}{z_t}\right)-1\right)\right] \end{aligned}$$

I simplified the calculation a bit: $(R+z)^2 \approx R^2$, because maximum z is about 100km, but R=6400km. Using the derived formula, we'll get mass of the atmosphere

$$M_a = 5.14 \cdot 10^{18}kg.$$

Compared to the mass of Earth, which is about $5.9 \cdot 10^{24}kg$, the mass of the atmosphere is about a million times smaller.

From the site http://www.eia.doe.gov/oiaf/ieo/emissions.html I got data of CO2 emission in 2003 and from the site http://www.eia.doe.gov/iea/carbon.html I got information about 2004. The annual total emission is about 27,000 million metric tons which is about $2.7 \cdot 10^{13}kg$ and it is about $100,000$ times smaller amount than the atmosphere is according to our calculations.

0.2 Problem #2

For self gravitational body we know that the equation of hydrostatic equilibrium is

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(\frac{r^2}{\rho}\frac{\partial P}{\partial r}\right) = -4\pi G\rho.$$

As we know that $P(\rho) = \frac{C}{2}\rho^2$, where $C = \frac{P_0}{\rho_0^2}$, we can write $\frac{\partial P}{\partial r}$ as

$$\frac{\partial P}{\partial r} = C\rho\frac{\partial \rho}{\partial r}.$$

By replacing this result into the initial equation and later considering the fact that $r = xR$, we'll get

$$\begin{aligned}
\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 C\frac{\partial\rho}{\partial r}\right) &= -4\pi G\rho, \\
\frac{1}{x^2R^2}\frac{\partial}{\partial(xR)}\left(x^2R^2\frac{\partial\rho}{\partial(xR)}\right) &= \frac{-4\pi G\rho}{C} \\
\frac{1}{x^2}\frac{\partial}{\partial x}\left(x^2\frac{\partial\rho}{\partial x}\right) &= -\frac{4\pi GR^2\rho}{C}
\end{aligned}$$

As we check the unit, we can see that the dimensionless variable, that controls the solution, is $\frac{R^2G}{C}$, because the unit of the variable is $\left[\frac{R^2G}{C}\right] = \frac{m^2m^3s^2kg}{kg\cdot s^2m^5} = 1$. For further calculations the constant $\frac{4\pi GR^2}{C}$ is denoted as k^2. So the equation is:

$$\frac{1}{x^2}\frac{\partial}{\partial x}\left(x^2\frac{\partial\rho}{\partial x}\right) = -k^2\rho. \tag{1}$$

0.2.1 Solving the equation.

Using the example of quantum mechanics, Schrödinger equation, where radial part was solved by replacing $\rho = \frac{u(x)}{x}$, we can also write the equation 1 as follows:

$$\begin{aligned}
\frac{1}{x^2}x\frac{\partial^2 u}{\partial x^2} &= -k^2\frac{u}{x} \\
\frac{\partial^2 u}{\partial x^2} &= -k^2 u,
\end{aligned} \tag{2}$$

because $\frac{\partial\rho}{\partial x} = \frac{x\frac{\partial u}{\partial x} - u}{x^2} \rightarrow \frac{\partial}{x}\left(x^2\frac{\partial\rho}{\partial x}\right) = x\frac{\partial^2 u}{\partial x^2}$.

General solution for Eq. 2 is $u = A\cdot exp(ikx) + B\cdot exp(-ikx)$. Considering the boundary conditions $u(1) = 0$, it could be written that $A\cdot exp(ik) + B\cdot exp(-ik) = 0 \rightarrow B = -A\cdot exp(2ik)$. Now we can write for u

$$\begin{aligned}
u &= Ae^{ikx} - Ae^{2ik-ikx} \\
u &= Ae^{ik}\left(e^{ikx-ik} - e^{ik-ikx}\right) \\
u &= 2iAe^{ik}\frac{\left(e^{ik(x-1)} - e^{-ik(x-1)}\right)}{2i} \\
\rho &= \frac{L}{x}sin\left(k(x-1)\right).
\end{aligned}$$

Now it is possible to find constant L from the relation $M = 4\pi R^3\int_0^1 x^2\rho(x)dx$

$$\begin{aligned}
M &= 4\pi R^3 L\int_0^1 x\cdot sin(k(x-1))dx \\
M &= 4\pi R^3 L\left(\frac{sin(k(x-1))}{k^2} - \frac{x\cdot cos(k(x-1))}{k}\right)_0^1
\end{aligned}$$

$$
\begin{aligned}
M &= 4\pi R^3 L\left(-\frac{1}{k}-\frac{sin(-k)}{k^2}\right)\\
M &= \frac{-4\pi R^3 L}{k}\left(1-sinc(k)\right).
\end{aligned}
$$

So the constant L would be

$$L = -\frac{kM}{4\pi R^3(1-sinc(k))},$$

and $\bar{\rho}$would be

$$\bar{\rho} = -\frac{3L}{k}(1-sinc(k)) = \frac{3M}{4\pi R^3}.$$

Now le'ts write down the ρ and analyze it a bit.

$$\rho = -\frac{kM\cdot sin(k(x-1))}{x\cdot 4\pi R^3(1-sinc(k))}.$$

To get a reasonable result, we have to assume that there are no oscillations in range of 0 to 1 for x. If k would be $k=[0,\pi]$, the term $(1-sinc(k))$ would be positive (and zero when k=0) and sin function would be definitely negative or zero. So the ρwould be positive and that's what we basically need. For example, the shape of the ρ is shown in Figure 1.

0.2.2 Figure of $\rho/\bar{\rho}$

First of all, let's find $\rho/\bar{\rho}$

$$\rho/\bar{\rho} = -\frac{k\cdot sin(k(x-1))}{x\cdot 3(1-sinc(k))}. \tag{3}$$

As we can see, $\bar{\rho}$ does not depend on x. So basically the plot of the Eq. 3 would look very similar to the plot in Figure 1. It's only matter of choosing constants

0.3 Linearizing the equations

Equations from Afternote.2.tex:

$$
\begin{aligned}
\frac{\partial\rho}{\partial t}+\nabla\cdot(\rho\vec{u}) &= 0, & (4)\\
\left(\frac{\partial}{\partial t}+\vec{u}\cdot\nabla\right)\vec{u} &= \frac{\vec{F}}{m}-\frac{1}{\rho}\nabla P+\frac{D_\eta}{3}\nabla(\nabla\cdot\vec{u})+D_\eta\nabla^2\vec{u}, & (5)\\
\left(\frac{\partial}{\partial t}+\vec{u}\cdot\nabla\right)\theta &= -\frac{1}{C_v}(\nabla\vec{u})\theta+D_\kappa\nabla^2\theta & (6)
\end{aligned}
$$

After linearizing with variables $\rho+\delta\rho$, $P_0+\delta P$, $T_0+\delta T$, $\delta\vec{u}\neq 0$ and considering Eq. 36 from Afternote.2.tex, which is $\delta P = \left(\frac{\partial P}{\partial\rho}\right)\delta\rho+\left(\frac{\partial P}{\partial\theta}\right)\delta\theta = c_0^2\delta\rho+V\delta\theta =$

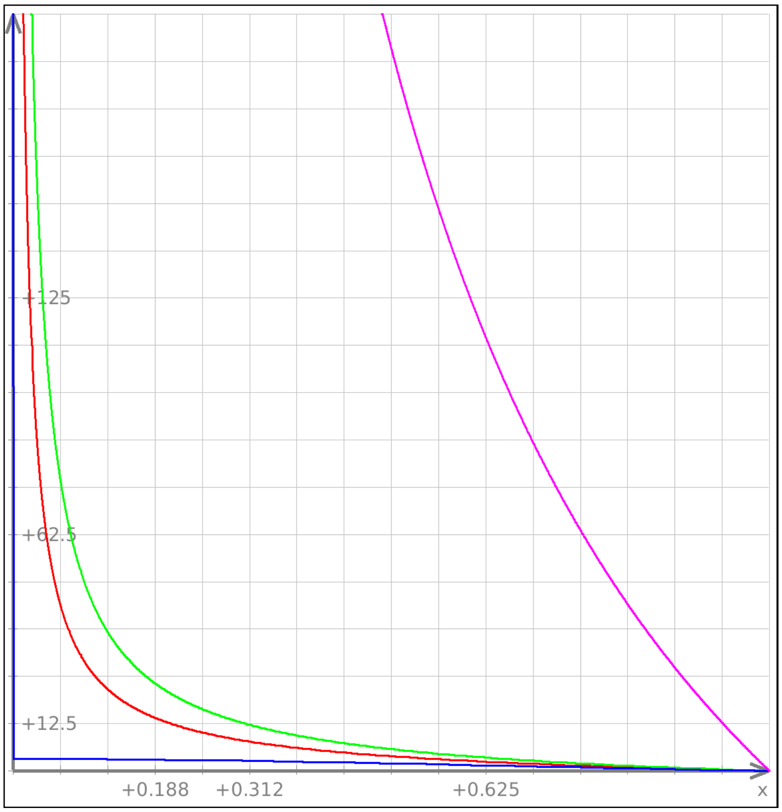

Figure 1: The shape of the ρ for different values of k. X axis is value of x. Y axis is the value of ρ. The blue color is for $k = \pi$, the red color is for $k = \frac{\pi}{2}$, the green color is for $k = \frac{\pi}{3}$, and the violet is for $k = \frac{\pi}{100}$.

$c_0^2\delta\rho + V\delta T$ approximately. The system becomes

$$\begin{aligned}
\frac{\partial \delta\rho}{\partial t} + \rho_0 \nabla \cdot \delta\vec{u} &= 0, \\
\frac{\partial \delta\vec{u}}{\partial t} &= -\frac{1}{\rho_0}\nabla\left(c_0^2\delta\rho + V\delta T\right) + \frac{D_\eta}{3}\nabla(\nabla\delta\vec{u}) + D_\eta\nabla^2\delta\vec{u}, \\
\frac{\partial \delta T}{\partial t} &= -\frac{T_0}{C_v}\nabla\delta\vec{u} + D_\kappa\nabla^2\delta T.
\end{aligned}$$

Next we'll insert wavelike disturbances to the equation:

$$\begin{aligned}
\delta\rho &= A_\rho e^{i(\vec{k}\vec{x}-wt)}, \\
\delta\vec{u} &= \vec{A}_u e^{i(\vec{k}\vec{x}-wt)}, \\
\delta T &= A_T e^{i(\vec{k}\vec{x}-wt)}.
\end{aligned}$$

After replacing we'll get the set of equations:

$$-i\omega A_\rho + \rho_0 i\vec{k}\vec{A}_u = 0 \Rightarrow A_\rho = \frac{\rho_0\vec{k}\vec{A}_u}{\omega} \tag{7}$$

$$-i\omega\vec{A}_u = -\frac{i}{\rho_0}\left(A_\rho\vec{k}c_0^2 + A_T V\vec{k}\right) - \frac{4D_\eta\vec{A}_u k^2}{3} \tag{8}$$

$$-i\omega A_T = -\frac{T_0 i\vec{k}\vec{A}_u}{C_v} - D_\kappa k^2 A_T \Rightarrow A_T = \frac{T_0 i\vec{k}\vec{A}_u}{C_v(i\omega - D_\kappa k^2)} \tag{9}$$

By inserting Eq. 7 and Eq. 9 into the Eq. 8, the result we'll get is

$$\begin{aligned}
-i\omega\vec{A}_u &= -\frac{i}{\rho_o}\left(\frac{\rho_0 k^2 c_0^2\vec{A}_u}{\omega} + \frac{T_0 i k^2 V\vec{A}_u}{C_v(i\omega - D_\kappa k^2)}\right) - \frac{4}{3}D_\eta k^2\vec{A}_u| \cdot \frac{i}{\vec{A}_u}, \\
\omega &= \frac{k^2 c_0^2}{\omega} + \frac{T_0 i k^2 V}{C_v i\omega\rho_0 - \kappa k^2} - \frac{4}{3}D_\eta i k^2, \\
\omega &= \frac{k^2}{\omega}\left(c_0^2 + \frac{T_0 i V}{C_v i\rho_0 - \kappa\frac{k^2}{\omega}}\right) - \frac{4k^2}{3}iD_\eta
\end{aligned}$$

if we denote $\frac{k^2}{\omega}$by X, the equation would look like little bit better (not as good as I expected actually..)

$$\begin{aligned}
\omega &= X\left(c_0^2 + \frac{T_0 i V}{C_v i\rho_0 - \kappa X}\right) - \frac{4k^2}{3}iD_\eta \\
\omega &= X\left(c_0^2 - \frac{T_0 i V \cdot (\kappa X + C_v\rho_0)}{C_v^2\rho_0^2 + \kappa^2 X^2}\right) - \frac{4k^2}{3}iD_\eta
\end{aligned}$$

considering all those constants in the equation, we can write little bit nicer form of the equation (without showing the constants explicitly)

$$\omega = X\left(c_0^2 - \frac{AX + B}{D + \kappa^2 X^2}\right) - Fk^2.$$

I tried also getting the relation using determinant or different methods. So far I haven't had success to get a nice and simple result.

About the (c). If I have understood correctly, then the mechanism are not additive. Though the units of those transport coefficients are the same, the mechanism (equations describing the attenuation) could not be added...