Physics 740: Spring 2006: 02/26/07

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z: uniform flow, source/sink, dipole, ... For D=2 problems it often occurs that a complex flow field can be built up from a set of rudimentary flows. The target here is the flow around a cylinder which is lifted by the Bernoulli force. Before we get to that here are some pieces. First recall notation and relationships

$$\mathbf{v} = (u, v, w) = (u, v, 0), \tag{1}$$

$$\mathbf{v} = \nabla\phi, \tag{2}$$

$$\mathbf{v} = \left(\frac{\partial\psi}{\partial y}, -\frac{\partial\psi}{\partial x}\right), \tag{3}$$

$$\nabla^2\phi = 0, \tag{4}$$

$$\nabla^2\psi = 0, \tag{5}$$

$$F(z) = \phi(x, y) + i\psi(x, y), \tag{6}$$

where ϕ is the *velocity potential*, ψ is the *stream function* and the boundary condition at fluid/solid interfaces is $\mathbf{n}\cdot\mathbf{v} = 0$.

1. uniform flow:

$$F(z) = v_0 z = v_0 x + i v_0 y, \tag{7}$$

$$\phi = v_0 x, \tag{8}$$

$$\psi = v_0 y, \tag{9}$$

$$\mathbf{v} = (v_0, 0). \tag{10}$$

$$\tag{11}$$

2. point source at the origin:

$$F(z) = v_0 a \ln z = v_0 a \ln re^{i\theta} = v_0 a \ln r + i v_0 a\theta, \tag{12}$$

$$\phi = v_0 a \ln r, \tag{13}$$

$$\psi = v_0 a\theta, \tag{14}$$

$$\mathbf{v} = v_0 \frac{a}{r}\, \mathbf{e}_r. \tag{15}$$

$$\tag{16}$$

3. circulation at the origin:

$$F(z) = -iv_0 a \ln z = -iv_0 a \ln re^{i\theta} = -iv_0 a \ln r + v_0 a\theta, \tag{17}$$

$$\phi = v_0 a\theta, \tag{18}$$

$$\psi = -v_0 a \ln r, \tag{19}$$

$$\mathbf{v} = v_0 \frac{a}{r}\, \mathbf{e}_\theta. \tag{20}$$

$$\tag{21}$$

4. *dipole* at the origin: this is the limit as $b \to 0$ of two point sources $\pm$ at $z_0 = (\pm b, 0)$,

$$F(z) = v_0 a \ln(z+b) - v_0 a \ln(z-b) \to 2v_0 a \frac{b}{z}, \tag{22}$$

$$\phi = 2v_0 ab\, \frac{x}{x^2+y^2}, \tag{23}$$

$$\psi = -2v_0 ab\, \frac{y}{x^2+y^2}. \tag{24}$$

The lines of constant stream function are circles centered on the y-axis

$$x^2 + \left(y + \frac{v_0 ab}{\psi}\right)^2 = \left(\frac{v_0 ab}{\psi}\right)^2. \tag{25}$$

5. cylinder of radius b in a uniform flow: combine dipole and uniform flow of appropriate strength,

$$F(z) = 2v_0 a \frac{b}{z} + v_0 a M z, \tag{26}$$

$$\phi = v_0 a \left(\frac{2b}{r} + Mr\right) \cos\theta, \tag{27}$$

$$\psi = v_0 a \left(-\frac{2b}{r} + Mr\right) \sin\theta. \tag{28}$$

Require $\psi = 0$ at $r = b$, i.e., $M = 2/b$. Thus

$$\phi = v_\infty b \left(\frac{b}{r} + \frac{r}{b}\right) \cos\theta, \tag{29}$$

$$\psi = v_\infty b \left(-\frac{b}{r} + \frac{r}{b}\right) \sin\theta, \tag{30}$$

where $v_\infty b = 2v_0 a$. From Eq. (29) find

$$v_r = v_\infty \left(1 - \frac{b^2}{r^2}\right) \cos\theta \asymp v_\infty \frac{x}{r}, \tag{31}$$

$$v_\theta = -v_\infty \left(1 + \frac{b^2}{r^2}\right) \sin\theta \asymp -v_\infty \frac{y}{r}. \tag{32}$$

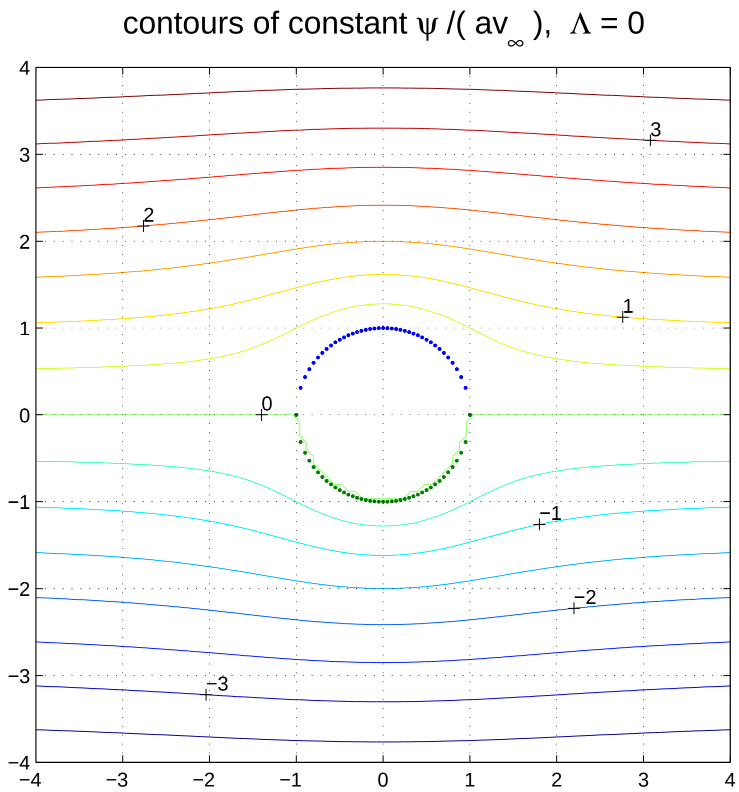

FIG. 1: Stream lines around a cylinder of radius 1, $a = b = 1$.

To calculate the pressure along any stream line use the Bernoulli equation in the form

$$P_0 + \frac{1}{2}\rho_0 v_\infty^2 = P + \frac{1}{2}\rho_0 v^2, \tag{33}$$

where $P_0 + \rho_0 v_\infty^2/2$ is appropriate for points on the streamline far from the cylinder. Then, $\delta P = P - P_0$ is given by

$$X = \frac{\delta P}{\frac{1}{2}\rho_0 v_\infty^2} = 1 - \frac{v^2}{v_\infty^2}. \tag{34}$$

Along the $\psi = 0$ streamline, $r = b$, $v_\theta = -2v_\infty \sin\theta$ and

$$X = 1 - 4\sin^2\theta. \tag{35}$$

The excess pressure is 1 at the stagnation points, $\theta = 0, \pi$ and -3 at $\theta = \pm\pi/2$. As X is symmetric in θ about $\theta = 0$, while there is pressure variation around the cylinder, there is no net force on it.

6. cylinder of radius $b = a$ in a uniform flow with an added circulation about the origin

$$F(z) = v_\infty a\left(\frac{a}{z} + \frac{z}{a}\right) + i\frac{\Gamma a}{2\pi}\ln\frac{z}{a}, \tag{36}$$

$$\phi = v_\infty a\left(\frac{a}{r} + \frac{r}{a}\right)\cos\theta - \frac{\Gamma a}{2\pi}\theta, \tag{37}$$

$$\psi = v_\infty a\left(-\frac{a}{r} + \frac{r}{a}\right)\sin\theta + \frac{\Gamma a}{2\pi}\ln\frac{r}{a}. \tag{38}$$

For the velocities find

$$v_r = v_\infty\left(1 - \frac{a^2}{r^2}\right)\cos\theta \tag{39}$$

$$v_\theta = -v_\infty\left(1 + \frac{a^2}{r^2}\right)\sin\theta - \frac{\Gamma}{2\pi}\frac{a}{r}. \tag{40}$$

Note $v_r = 0$ at $r = a$ and is unchanged from above. Scale the velocities by v_∞, define $\Lambda = \Gamma/(2\pi v_\infty)$ and find

$$\frac{v_r}{v_\infty} = \left(1 - \frac{a^2}{r^2}\right)\cos\theta \tag{41}$$

$$\frac{v_\theta}{v_\infty} = -\left(1 + \frac{a^2}{r^2}\right)\sin\theta - \Lambda\frac{a}{r}. \tag{42}$$

Calculate the pressure as above. Along the $\psi = 0$ streamline the symmetry of v_θ, $v(\theta) = v(-\theta)$, is *broken* by the circulation, Λ. [A visual manifestation of this is the

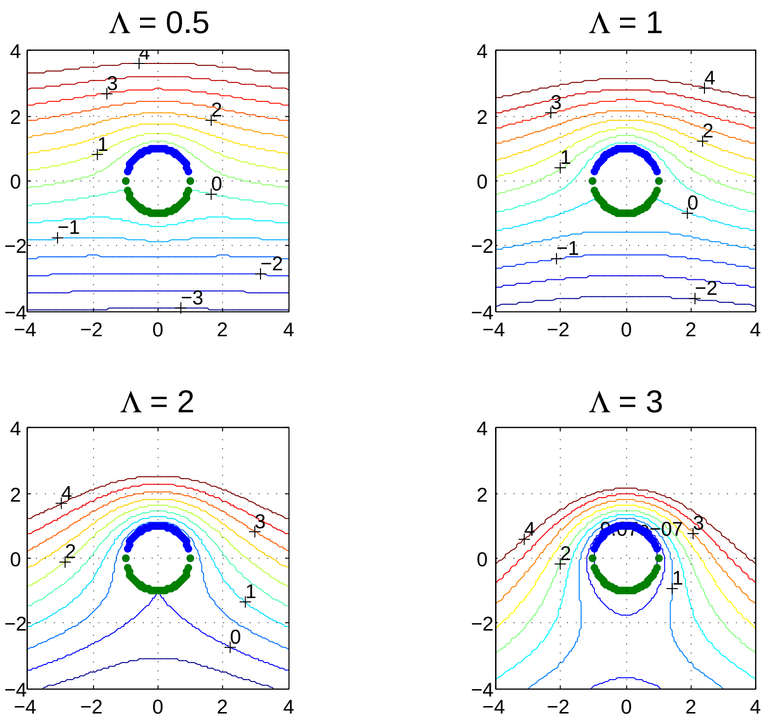

FIG. 2: Stream lines around cylinder for various values of the circulation.

change in the location of the stagnation points from $0, \pi$, e.g., Fig. 2.] Along the $\psi = 0$ streamline

$$X = 1 - 4\sin^2\theta - \Lambda^2 - 4\Lambda\sin\theta. \tag{43}$$

The part of X that is symmetric in θ contributes nothing to the pressure difference between the top and bottom of the cylinder. It is the last term that counts. The net force, in the y-direction, due to the pressure is given by

$$F_y = \int_0^{2\pi} d\theta(-\sin\theta)(-4\Lambda\sin\theta) = 4\pi\Lambda. \tag{44}$$

Because v_θ is symmetric about $\theta = \pi/2$ there is no net force in the x-direction,

$$F_x = 0. \tag{45}$$