Physics 740: Spring 2006:

Afternote.3.tex 02/28/07

Complex Variables and Fluid Flows in D=2.

1. Complex Variables. For $F(z) = u(x, y) + iv(x, y)$

$$\frac{\partial F}{\partial x} = \frac{dF}{dz}\frac{\partial z}{\partial x} = \frac{dF}{dz}, \tag{1}$$
$$\frac{\partial F}{\partial y} = \frac{dF}{dz}\frac{\partial z}{\partial y} = i\frac{dF}{dz}. \tag{2}$$

Then using $F(z) = u + iv$

$$\frac{dF}{dz} = \frac{\partial F}{\partial x} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}, \tag{3}$$
$$\frac{dF}{dz} = -i\frac{\partial F}{\partial y} = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}. \tag{4}$$

Thus

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \tag{5}$$
$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}, \tag{6}$$

the Cauchy-Riemann conditions. These conditions are obeyed by any analytic function of $z = x + iy$. Thus for any analytic function

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial x}\frac{\partial v}{\partial y} + \frac{\partial}{\partial y}\left(-\frac{\partial v}{\partial x}\right) = 0 \tag{7}$$

and $\nabla^2 v = 0$.

2. Fluid Flow. For an ideal fluid in two dimensions the the velocity field $\mathbf{v}(x, y) = (u(x, y), v(x, y), 0)$ must obey the eqautions

$$\nabla \times \mathbf{v} = 0, \tag{8}$$
$$\nabla \cdot \mathbf{v} = 0, \tag{9}$$

pointwise throughout the fluid and satisfy the BC $\mathbf{n} \cdot \mathbf{v} = 0$ on all interfaces.

1. If **v** is found from a velocity potential

$$\mathbf{v} = \nabla\phi, \tag{10}$$

then **v** automatically satisfies Eq. (7) and Eq. (8) implies $\nabla^2\phi = 0$.

2. If **v** is found from the stream function according to

$$\mathbf{v} = (u, v) = \left(\frac{\partial\psi}{\partial y}, -\frac{\partial\psi}{\partial x}\right), \tag{11}$$

then **v** automatically satisfies Eq. (8) and Eq. (7) implies $\nabla^2\psi = 0$.

3. So ϕ and ψ satisfy the same set of equations as the two components of an analytic function. So associate ϕ and ψ with u and v. Thus *any analytic function of z is in principle a possible two dimensional fluid flow.*

4. Be aware that you have to be slightly careful if you are dealing with a multiply connected space. See Land L page 16 at the bottom.

3. Generalization. Complex variable methods are used extensively in solving D=2 E&M problems. But not Jackson. See page 79 of the 2^{nd} edition of Jackson for reference to **complex variables** and **conformal mapping**, i.e., the end of the second chapter. Mapping of a complicate space into a simple space is an important feature of D=2 complex variable methods. The Schwarz-Christoffel transformation arises in this context. MATLAB has a **Schwarz-Christoffel Toolbox**.