Physics 740: Spring 2006: 02/14/07

P740.6.tex

Ideal Fluids. The equations for an ideal fluid are the continuity equation, the Euler equation and an equation for the conservation of energy:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0, \quad (LL\ 1.2) \tag{1}$$

$$\left(\frac{\partial}{\partial t} + \mathbf{u}\cdot\nabla\right)\mathbf{u} = \frac{\mathbf{F}}{m} - \frac{1}{\rho}\nabla P, \quad (Euler) \quad (LL\ 2.9) \tag{2}$$

$$\left(\frac{\partial}{\partial t} + \mathbf{u}\cdot\nabla\right)\theta = -\frac{2}{3}(\nabla\cdot\mathbf{u})\theta, \quad (LL\ 6.1). \tag{3}$$

The equations in LL differ slightly from these as you will explain in the third problem set. For the ideal fluid there is no viscosity and no thermal transport, $D_\eta \propto \eta = 0$ and $D_\kappa \propto \kappa = 0$. Since $\mathbf{Q} = 0$ all processes take place at constant entropy.

Hydrostatic Equilibrium. The equation of hydrostatic equilibrium comes from the Euler equation upon putting $\mathbf{u} = 0$ [It is also found from the Navier-Stokes equation under the same conditions]

$$\mathbf{f} - \frac{1}{\rho}\nabla P = 0, \tag{4}$$

where $\mathbf{f} = \mathbf{F}/m$ is the force per unit mass.

(a) For the case of a self gravitating object with spherical symmetry this equation takes the form

$$\left(\frac{r^2}{\rho}\frac{\partial P}{\partial r}\right) = -G\int_0^r 4\pi r^2 \rho(r) dr, \tag{5}$$

or

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(\frac{r^2}{\rho}\frac{\partial P}{\partial r}\right) = -4\pi G\rho. \tag{6}$$

(b) If the density of the object drops markedly at $r = R$ the remaining material at $r > R$ can be described by

$$\left(\frac{r^2}{\rho}\frac{\partial P}{\partial r}\right) = -G\int_0^R 4\pi r^2 \rho(r) dr = -GM. \tag{7}$$

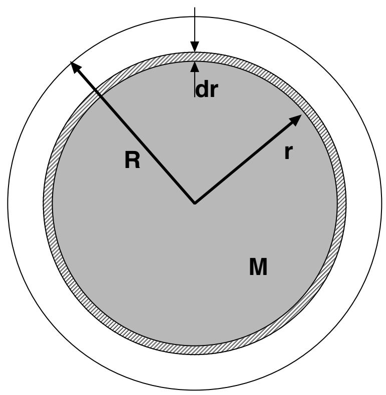

FIG. 1: Astrophysical object. $P(R) = 0$, The shell at r is pulled toward the center by a gravitational force proportional to the mass $M(r) = 4\pi \int_0^r r^2 \rho(r) dr$. The pressure gradient across the shell, proportional to dP/dr, balances the gravitational force. If it can.

Stable to Convective Motion. The situation is as described in Fig. 2. A cell of air is moved upward by dz. It is unstable in its new location if the volume it occupies is less than the volume of the cells it joins. (All cells have the same number of particles, the same mass.) A cell with smaller volume is more dense and is pulled downward relative to its neighbors. For the cell from below to not want to convect upward require

$$V(P^{'}, S^{'}) - V(P^{'}, S) = \left(\frac{\partial V}{\partial S} \right)_P \frac{dS}{dz} > 0. \tag{8}$$

The cell of air moved upward takes its S along as there is **NO** method of energy transfer. Since $(\partial V / \partial S)_P > 0$ it must be that for no convection

$$\frac{dS}{dz} > 0. \tag{9}$$

See details in *LL sec.*1.4. Here is a simple argument (not as general as it uses the ideal

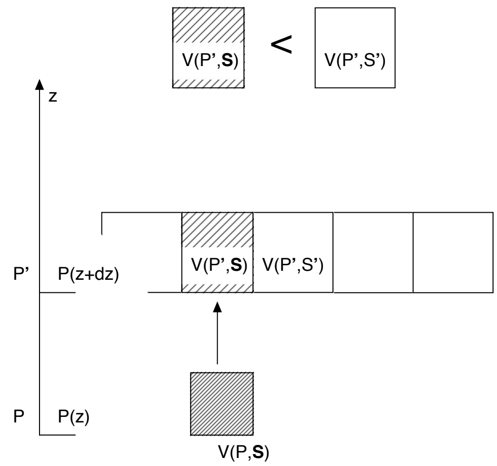

FIG. 2: A cell of air containing N particles at (P, S, z) moves to $(P^{'}, S, z+dz)$ where it has volume $V(P^{'}, S)$. At $z+dz$ the cell that moved is among cells of N particles that have volume $V(P^{'}, S^{'})$. The cell that moved will be more dense than the cells it joined if $V(P^{'}, S), V(P^{'}, S^{'})$. So gravity will tend to pull it downward.

gas as model). The entropy per particle of an ideal gas of N particles is

$$s = \frac{S}{N} = k_B \; ln \left(\frac{V}{N} \frac{1}{\lambda_T^3} \right), \tag{10}$$

where λ_T, the thermal DeBroglie wavelength is $\lambda_T = \hbar/p_T$ and $p_T^2 \approx m k_B T$. To express V/N in terms of P and T use the ideal gas law. Then

$$s = \frac{S}{N} = k_B \; ln \left(\frac{T^{5/2}}{P} \cdots \right) \approx k_B \; (\frac{5}{2} lnT - lnP) \tag{11}$$

and

$$\frac{ds}{dz} = k_B \; \left(\frac{5}{2} \frac{dT}{dz} - \frac{dP}{dz} \right). \tag{12}$$

Note that T and P drive the entropy in opposite directions. Understand this. Agree that both T and P decrease as z increases so that we have $ds/dz > 0$ for

$$\left| \frac{dT}{dz} \right| < \frac{2}{5} \left| \frac{dP}{dz} \right|. \tag{13}$$

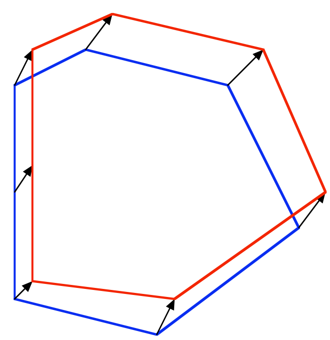

FIG. 3: A contour in the fluid, initially blue, moves with the local fluid velocity to a new location, red, where it is deformed and where the velocity at each point on the contour has changed.

Compare to *LL* (4.5)

Circulation and Vorticity. The circulation, Γ, is defined to be the integral around a **closed contour** in the fluid that moves with the fluid. As such it is deformed over time as the fluid moves, Fig. 3.

$$\Gamma = \int_{\mathbf{C}} \mathbf{u} \cdot d\mathbf{r}. \tag{14}$$

The rate of change of the circulation has two parts, one from the change in the velocity at each point on the contour and a second from the change in the contour as it moves with the fluid, see Fig. 4,

$$\frac{d\Gamma}{dt} = \int_{\mathbf{C}} \frac{d\mathbf{u}}{dt} \cdot d\mathbf{r} + \int_{\mathbf{C}} \mathbf{u} \cdot \frac{d(d\mathbf{r})}{dt}. \tag{15}$$

1. Using $d(d\mathbf{r})/dt = d\mathbf{u}$ the second term is the integral of an exact differential, $\int_{\mathbf{C}} d(u^2/2)$, around a closed contour and equal to zero.

2. Use

$$\frac{d\mathbf{u}}{dt} = \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\frac{1}{\rho} \nabla P = -\nabla h, \tag{16}$$

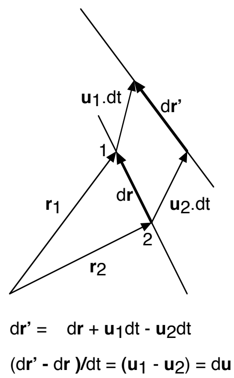

FIG. 4: The change in the line segment $d\mathbf{r}$ in time dt is due to the motion of the segment with the local fluid velocity. Thus $d(d\mathbf{r})/dt = d\mathbf{u}$.

where $h = w$ is the enthalpy (see the discussion above *LL* 2.9). Then the first term is $-\int_{\mathbf{C}} dh$, an exact differential, and also equal to zero.

Thus $d\Gamma/dt = 0$, the circulation around a contour in the fluid is constant over time or possibly zero.

When Γ is calculated around an infinitesimal closed contour you can write

$$\delta\Gamma = \int_{\delta\mathbf{C}} \mathbf{u} \cdot d\mathbf{r} = \int_{\delta\mathbf{S}} \nabla \times \mathbf{u} \cdot \delta\mathbf{S} \approx (\nabla \times \mathbf{u}) \cdot \delta\mathbf{S} = constant. \tag{17}$$

The quantity $\nabla \times \mathbf{u}$ at a point in the fluid is the vorticity at that point. For the vorticity to be constant it *must move with the fluid.*