P740.HW5.sol.tex

1. Wedge shaped trough. Since the depth as a function of position is $h(x) = h_0(1-x/a)$ the basic equation to be solved becomes

$$\delta\ddot{h} = \frac{\partial}{\partial x}\left[gh_0\left(1-\frac{x}{a}\right)\frac{\partial\delta h}{\partial x}\right]. \tag{1}$$

Solution to the ODE. As you are looking for the normal modes use $\delta h = H(x)\ exp(-i\omega t)$ and to sanitize the resulting equation $z = x/a$, $c_0^2 = gh_0$, $\tau = c_0 t/a$ and $\Omega = \omega a/c_0$. Find

$$(1-z)H^{''} - H^{'} + \Omega^2 H = 0, \tag{2}$$

where $^{'} = d/dz$. Make the change of variable $\zeta = 1 - z$ then,

$$H^{''} + \frac{1}{\zeta}H^{'} + \frac{\Omega^2}{\zeta}H = 0. \tag{3}$$

where now $^{'} = d/d\zeta$. This equation has solution as a Bessel function (e.g., Boas 12.16.1)

$$H = AJ_0(2\Omega\zeta^{\frac{1}{2}}). \tag{4}$$

Modes. At the edge of the trough $z = 1$ and $\zeta = 0$. At the trough center $z = 0$ and $\zeta = 1$. There are two types of modes,

1. spatially odd modes with a node at $\zeta = 1$ (trough center), i.e., modes for which 2Ω is a zero of the J_0 Bessel function, $\Omega_n = z_0^{(n)}/2$.

2. spatially even modes around the trough center. For these modes you need $dJ_0(2\Omega\zeta^{\frac{1}{2}})/d\zeta \propto J_1(2\Omega\zeta^{\frac{1}{2}})$=0, i.e., modes for which 2Ω is a zero of the J_1 Bessel function, $\Omega_n = z_1^{(n)}/2$.

The zeros of the Bessel functions are tabulated. The results are shown in Figs. 1 and 2, note the values of Ω in the caption. These frequencies are sensibly related to the time for a disturbance with velocity c_0 to cross the distance $2a$. Recall $\omega = \Omega c_0/a$.

2. Interface of 2 fluids. Solve Laplaces equation in the two spaces;
space 1: $-h_1 < z < 0$, $\phi_1 = A\ coshk(h_1 + z)\ coskx\ cos\omega t$,
space 2: $0 < z < h_2$, $\phi_2 = B\ coshk(h_2 - z)\ coskx\ cos\omega t$.
At the interface in space 1 the Bernoulli equation is

$$\frac{\partial\phi_1}{\partial t} + g\zeta + \frac{1}{\rho_1}P_1 = 0. \tag{5}$$

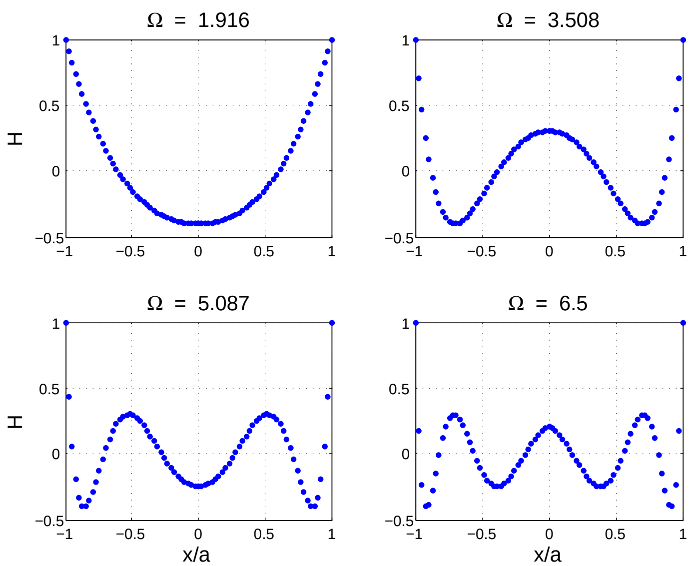

FIG. 1: Spatially even modes.

At the interface in space 2 the Bernoulli equation is

$$\frac{\partial \phi_2}{\partial t} + g\zeta + \frac{1}{\rho_2} P_2 = 0. \tag{6}$$

But the pressure must be continuous (in the absence of something at the interface that can exert force). Thus $P_1 = P_2$ and

$$g(\rho_1 - \rho_2)\zeta = \rho_2 \frac{\partial \phi_2}{\partial t} - \rho_1 \frac{\partial \phi_1}{\partial t}. \tag{7}$$

At the interface we also have $\dot{\zeta} = v_{1z} = v_{2z}$. From the expressions above for ϕ_1, ϕ_2 this means $A\ sinhkh_1 = -B\ sinhkh_2$. Taking the time derivative of Eq. (7) and using $\dot{\zeta} = \partial\phi_1/\partial z$

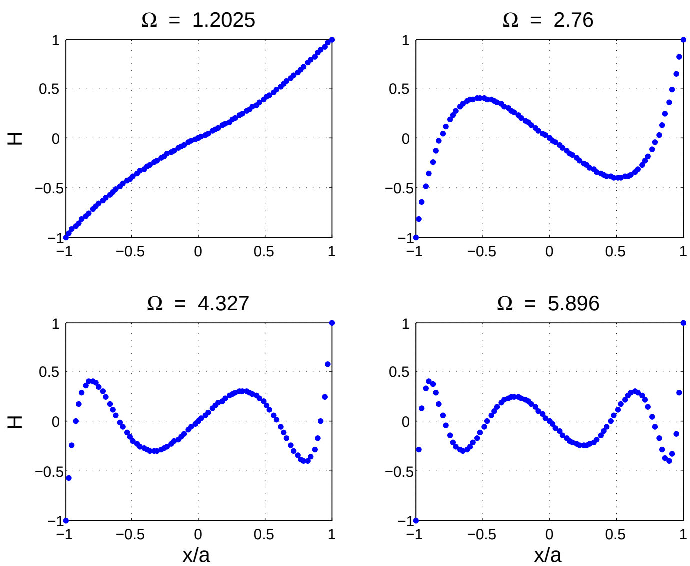

FIG. 2: Spatially odd modes.

leads to

$$g(\rho_1 - \rho_2)\frac{\partial \phi_1}{\partial z} = \rho_2 \frac{\partial^2 \phi_2}{\partial t^2} - \rho_1 \frac{\partial^2 \phi_1}{\partial t^2}. \tag{8}$$

Substitute and find

$$\omega^2 = \frac{gk(\rho_1 - \rho_2)}{\rho_1 \; cothkh_1 + \rho_2 \; cothkh_2} \tag{9}$$

In the limit $kh_1 \ll 1$ and $kh_2 \ll 1$ this reduces to

$$\omega^2 = \frac{gk^2(\rho_1 - \rho_2)h_1 h_2}{\rho_1 h_2 + \rho_2 h_1} \tag{10}$$

For $\rho_2 \rightarrow 0$ this goes over to the shallow water wave result

$$\omega^2 = gh_1 k^2. \tag{11}$$

3. Fourier transform. For $S(t)$ the Fourier transform is

$$S(\omega) = \int_{-\infty}^{+\infty} dt\ S(t) e^{i\omega t}. \tag{12}$$

For the case at hand the function S depends on $t - t_0$. So shift the origin of the t integration to t_0. This produces the factor $exp - \omega t_0$. The remaining integrals can be brought to the form

$$I_G = \int_{-\infty}^{+\infty} dx\ e^{-x^2+2Ax}, \tag{13}$$

where A is a collection of constants. This integral is done by completing the square in the argument of the exponential

$$I_G = e^{A^2} \int_{-\infty}^{+\infty} dx\ e^{-(x-A)^2} = \sqrt{\pi}\ e^{A^2}, \tag{14}$$

where the Gaussian integral is done by shifting the origin to A. Find

$$S(\omega) = e^{i\omega t_0} \frac{1}{2i} \left(e^{-\frac{1}{2}(\Delta t)^2(\omega+\Omega)^2} - e^{-\frac{1}{2}(\Delta t)^2(\omega-\Omega)^2} \right). \tag{15}$$

4. Dispersion. See the MATLAB listing **dispersion** in a separate pdf file. The result is shown in Fig. 3.

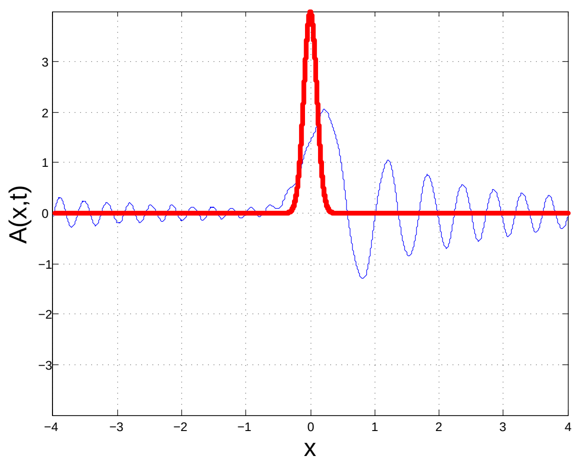

FIG. 3: The dispersing pulse has returned to the origin where it is compared to its initial form, the compact gaussian, heavy line.