Physics 740: Exam 2: 05/07/07

Two Problems. Due 8AM 05/07/07. **See note on page 8.**

Problem 1. An Artery for Bubba? The velocity of blood flow in an artery, modeled as the flow between two parallel plates of separation $2R$ and width $b \gg 2R$ (R is called the radius), obeys the equation of motion

$$\frac{\partial v_x}{\partial t} = -\frac{v_x}{\tau} + D_\eta \frac{\partial^2 v_x}{\partial y^2} - \frac{1}{\rho_0}\frac{\partial P}{\partial x}, \tag{1}$$

where $D_\eta = \eta/\rho_0$ and $P(x) = -(\Delta P/L)x$. Without the first term on the RHS this is an equation you know (Navier-Stokes for an incompressible fluid). You are interested in steady flow ($\partial/\partial t = 0$) solutions to this slightly more general equation under a variety of circumstances. An important ingredient of those circumstances will be the influence of fatty acid (FA) deposits (see below) on the blood flow through the artery. Think Bubba, 3 Big Macs a day, FA concentration in the blood of n_F. The extra term in Eq. (1) comes from the impedance to fluid flow caused by occasional long chains of FA that cross the artery, Fig. 1; they may appear to block the flow but remember the second dimension.

1. For fixed artery radius R_0 solve Eq. (1) for $v_x(y)$ when $\tau \to \infty$. Take the boundary condition on the flow at $y = \pm R_0$ to be $v_x(\pm R_0) = 0$. From your solution calculate the the mass current
$$Q = \rho_0\ b \int_{-R_0}^{+R_0} dy\ v_x(y). \tag{2}$$
2. Now go to the opposite limit. Set $D_\eta = 0$ at finite τ. For fixed artery radius R_0 solve Eq. (1) for $v_x(y)$. In this limit it will not be possible to implement the boundary condition $v_x(\pm R_0) = 0$ because there is no way to have v_x vary with y. Assume $v_x(y)$ goes to zero over a very short distances near $\pm R_0$ (perhaps on the molecular scale) so that you can ignore correction due to this. From your solution calculate the the mass current as above.

Before going on put the answer to these two questions in ohms law form. Ohms law is

$$I = \frac{1}{R}V, \tag{3}$$

where the voltage difference V causes a current I determined by a coefficient R, called the resistance, that depends on the geometry of the resistor and on intrinsic properties of the

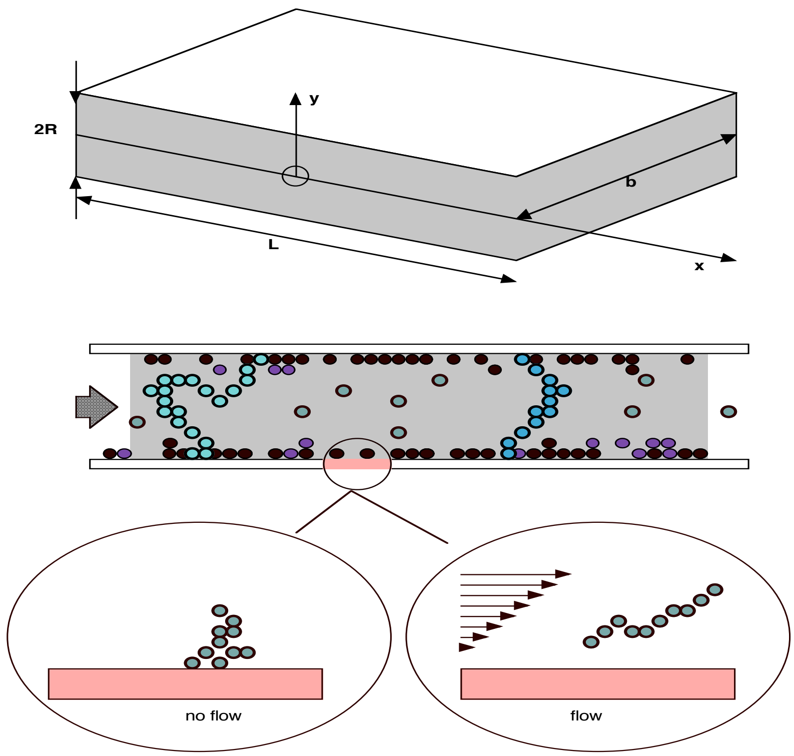

FIG. 1: Model for an artery.

stuff the resistor is made of. There is the correspondence $\Delta P \leftrightarrow V$, $Q \leftrightarrow I$ and

$$Q = \frac{1}{R_A}\Delta P, \tag{4}$$

where R_A is the resistance to mass flow of the artery. Calculate R_A for the the two cases above. Call the results R_A^η and R_A^τ for cases 1 and 2 respectively.

For fixed artery radius R_0 solve Eq. (1) for $v_x(y)$ in the general case. Take the boundary condition on the flow at $y = \pm R_0$ to be $v_x(\pm R_0) = 0$. From your solution calculate the the mass current from Eq. (2).

1. From the expression for Q calculate R_A. Put your answer in the form

$$R_A = R_A^\tau f(z), \tag{5}$$

where $z^2 = \kappa^2 R_0^2$ and $\kappa^2 = 1/(\tau D_\eta)$.

2. Show that when $z \to \infty$ you recover R_A^τ.

3. Show that when $z \to 0$ you recover R_A^η.

4. Explain why z (or z^2) is the controlling physical variable.

5. Plot $f(z)$ as a function of z.

Fatty acids not only cause τ they also attach to the walls of the artery, reducing its size, Fig. 1. As they do this they change the flow through the artery. This may not be all bad? The shear stress near the walls of the artery tends to liberate attached FA. Here is a model for the effect

$$\frac{dR}{dt} = -\frac{\sigma_0}{|\sigma_{xy}(R)|}\frac{1}{\kappa}\frac{1}{\tau_A}, \tag{6}$$

where the shear stress at the artery wall is

$$\sigma_{xy}(R) = -\eta\left(\frac{\partial v_x}{\partial y}\right)_{y=R} \tag{7}$$

$\sigma_0 = P_0/(\kappa L)$ and P_0 and τ_A are known parameters.

1. From your solution for $v_x(y)$ for generic radius R find the shear stress $\sigma_{xy}(R)$.

2. Plot $(\Delta P \sigma_0)/(P_0|\sigma_{xy}(R)|)$ as a function of κR,

3. Use $\sigma_0/|\sigma_{xy}(R)|$ in Eq. (6) and find R as a function of time.

4. Plot R as a function of time in the form R/R_0 vs t/τ_F, $\tau_F = \tau_A(\Delta P/P_0)$ for several values of κR_0.

5. Use your result for $R(t)$ to find $Q(t)$.

6. Plot Q as a function of time in the form $Q(R(t))/Q(R_0)$ vs t/τ_F for several values of κR_0.

Note:

(a) In almost all instances R appears in combination with κ. The meaningful variable is κR not simply R.

(b) Some numbers

1. Consider 0.1 cm $< R_0 <$ 1 cm.

2. $D_\eta = 0.05$ cm^2/sec, $\tau = 5$ sec.

3. $a = 10^{-7}$ cm, $\tau_A(0) = 10$ sec

$$\frac{1}{\tau_A} = \frac{1}{\tau_A(0)} \left(\frac{n_F}{n_F^c} - 1 \right), \tag{8}$$

   where

$$n_F = n_F^c \; exp(0.43 \; r_{BM}), \tag{9}$$

   and r_{BM} is the rate of Big Mac consumption in units per day.

4. $\Delta P = 0.5 P_0$.

Estimate the time t_B for Bubba's artery to block.

Suppose n_F depends on x. Consider two segments of an artery. One segment begins to block. At fixed ΔP across the entire artery how does this blockage effect the rate of blockage of the other segment. You might want to think of a pair of resistors in series with a constant voltage across the pair.

Problem 2. How to find a Submarine? The Navy has made extensive studies of the shallow wave, normal modes of the *submarine free* ocean, described by the equation

$$\frac{\partial^2 \delta h}{\partial t^2} = g \frac{\partial}{\partial x}\left(h_0(x) \frac{\partial \delta h}{\partial x} \right), \tag{10}$$

where

$$h_0(x) = h_0 \left(1 - \frac{x^2}{a^2} \right), \quad - a \le x \le +a. \tag{11}$$

When a submarine is trying to hide on the bottom of the ocean, say at $x = b = pa$, a defect in $h_0(x)$ is created

$$h_0(x) = h_0 \left(1 - \frac{x^2}{a^2} \right) + R^2 \delta(x - b), \quad - a \le x \le +a. \tag{12}$$

This defect disturbs the normal modes of the ocean and allows the location of the submarine to be determined. [The problem as stated here is a variation on a famous mathematical physics problem set by Mark Kac in a paper entitled "Can you hear the shape of a drum".]

1. Find the normal mode frequencies of the submarine free ocean, Eqs. (10) and (11).

   (a) Be sure to use $z = x/a$.

   (b) Scale ω^2 by $\omega_0^2 = c_0^2/a^2$, $c_0^2 = g h_0$, $\Omega^2 = \omega^2/\omega_0^2$.

   (c) The normal modes should be described by a well known orthonormal set of functions that are polynomials. Call these functions $\phi_n(x)$. $n = 1, 2, \cdots$. [If you do not find this let me know.]

2. Determine the shift in the normal mode frequencies when the ocean depth profile is disturbed as in Eq. (12). To do this use the fact that $R^2/(a h_0) \ll 1$ so that the effect of the submarine can be treated using perturbation theory. [This is the same as time independent PT problem from quantum mechanics. If you do not know TIPT proceed as described below.]

3. The Navy flies an earth satellite over the ocean once every 2 hours. The satellite measures the frequency of the normal modes, (use the Doppler shift caused by surface motion) usually the lowest 6 to 10 (useful) modes. Based on the frequency shift of these modes they can track the motion of the submarine.

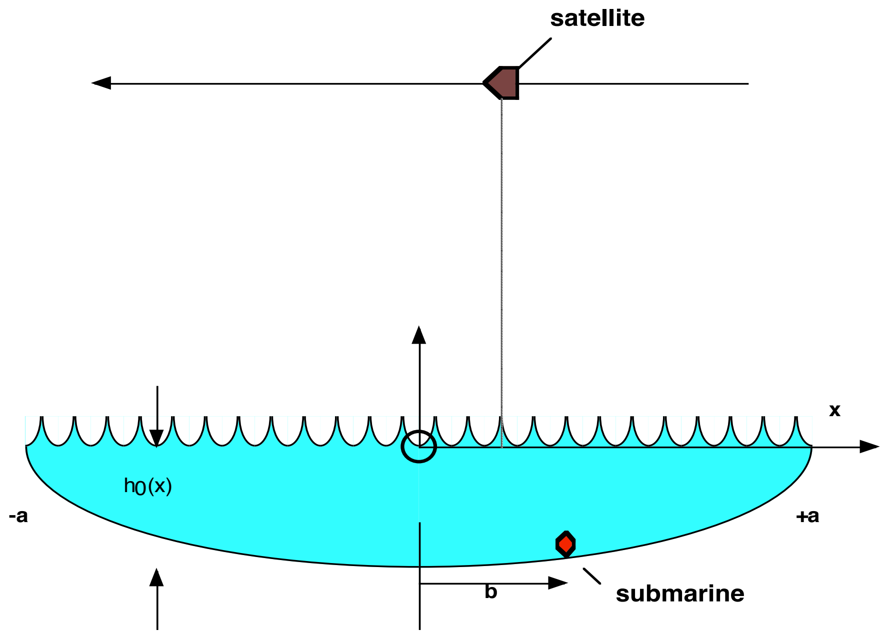

FIG. 2: Submarine at the bottom of the ocean.

4. From the data shown in Fig. 3 determine the location of the submarine as a function of time. [Hint 1. When a fly walks on a drumhead he will modify the frequencies of the normal modes. A skilled musician can get an idea about where the fly is by listening. When the fly steps on the node of a normal mode the mode *rings true* (otherwise not). Something qualitatively similar happens here.] [Hint 2. There are 6 modes in the figure. Which are they?]

5. Determine the color of the submarine.

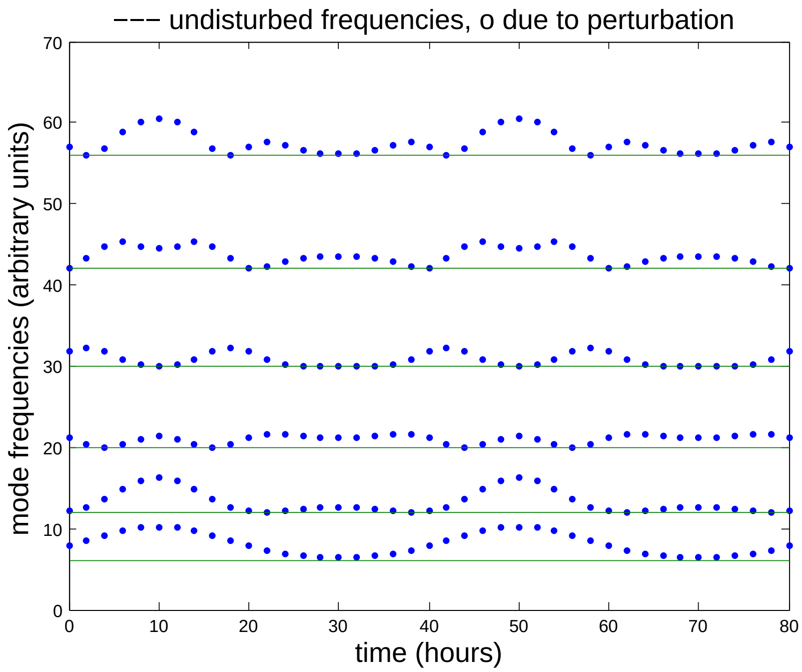

FIG. 3: Modes of the ocean as a function of time. When submarine is absent (solid lines), when present (circles).

TIPT

1. The normal modes solve the problem

$$-\Omega_n^2 \phi_n(z) = \frac{\partial}{\partial z} \left(\left[1 - z^2\right] \frac{\partial \phi_n(z)}{\partial z} \right) \tag{13}$$

with Ω_n^2 and $\phi_n(z)$ known.

2. When Eq. (12) is used for the depth profile Ω^2 and $\phi_n(z)$ change from the values in Eq. (13). Find

$$
\begin{aligned}
-\nu_n^2\theta_n(z) &= \frac{\partial}{\partial z}\left(\left[1-z^2\right]\frac{\partial\theta_n(z)}{\partial z}\right)+\epsilon\frac{\partial}{\partial z}\left(\delta(z-p)\frac{\partial\theta_n(z)}{\partial z}\right), \qquad (14)\\
&= K_0(z)\theta_n(z)+\epsilon K_1(z)\theta_n(z), \qquad (15)
\end{aligned}
$$

where $\epsilon = R^2/(ah_0)$. For $\epsilon \to 0$, $\nu_n^2 \to \Omega_n^2$ and $\phi_n \to \theta_n$. Thus write

$$
\begin{aligned}
\nu_n^2 &= \Omega_n^2+\epsilon\alpha_n+\epsilon^2\beta_n+\cdots, \qquad (16)\\
\theta_n(z) &= \phi_n(z)+\epsilon f_n(z)+\epsilon^2 g_n(z)+\cdots. \qquad (17)
\end{aligned}
$$

Substitute these expressions into Eq. (14) and find the coefficient of like powers of ϵ. These coefficients are set to zero producing a set of equations involving α_n, β_n, $\cdots$ $f_n(z)$, $g_n(z)$, $\cdots$ etc. You want the leading frequency shift α_n. This is found in the coefficient of ϵ which produces

$$
-\alpha_n\phi_n+(K_0(z)-\Omega_n^2)f_n=\frac{\partial}{\partial z}\left(\delta(z-p)\frac{\partial\phi_n(z)}{\partial z}\right), \qquad (18)
$$

3. Solve for α_n. You need two things to do this; (i) the functions $\phi_n(z)$ are orthogonal to one another and (ii) $f_n(z)$ does not contain ϕ_n. Multiply by ϕ_n and integrate over the domain of the functions. The equation you find for α_n should be sensitive to $\partial\phi_n/\partial z$ at $z = p$.

Note. Submit your work in 2 parts. Part 1 is a summary sheet, e.g., "for this question I found the answer lah-lah", ... , figures, ... Part 2 contains the details — particularly if you are not certain of your answer. In Part 1 cite the place in Part 2 where each answer was found.