Analytical Solution for Voltage-Step Response of Lossy Distributed RC Lines

Andres Punning and Erko Jalviste

Abstract— **This paper presents the analytical solution for the voltage-step response of finite-length open-ended distributed lossy RC lines. Two distributed conductivities of different nature causing the transversal charge loss are considered. Telegrapher's equation for voltage is solved at three different initial/boundary conditions using the method of Fourier series.**

Index Terms — **Distributed parameter circuits, Lossy circuits, RC circuits, Transient response**

I. INTRODUCTION

THE DISTRIBUTED RC LINES are often encountered in diverse fields of electronics. They can be used to model wires inside integrated CMOS chips [1], diodes [2] or thin-film capacitors [3]. The lossy RC line model can be used to describe even systems far from the traditional scope of electronics, for example neural transmission delays in neurophysiology [4] or electrochemical equivalents of electroactive polymers [5]-[6].

In the current paper we consider a distributed line composed of a dielectric layer sandwiched between the ground plate and a conductive layer. We take into account two distributed parameters of the dielectric layer: the capacitance and the loss in dielectric owing to its residual conductivity. The possible configurations of the lines are presented in Fig. 1. In case the dielectric layer is an ideal isolator, the line is fully described by the resistance of the conductive layer R and the capacitance C of the dielectric as shown in Fig. 1-A. This configuration is usually called a "lossless RC line" or simply a "RC line". The line depicted in Fig. 1-C incorporates two factors of loss of the dielectric - the conductance G of the dielectric layer, and the conductance W between the conductive layer and the capacitance of the dielectric. A circuit with the conductance G only is presented in Fig. 1-B. Hereinafter we refer to these configurations as the "RCGW line" and the "RCG line" respectively.

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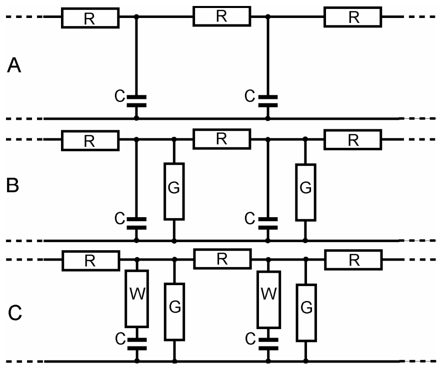

Fig.1. Three kinds of the distributed RC lines represented by a series equivalent circuits with discrete elements: (A) the lossless RC line; (B) the RCG line and (C) the RCGW line.

Transients of voltage and current in transmission lines are commonly treated by the Telegrapher's equations - a pair of partial differential equations (PDE) in two variables, the time and the coordinate [7].

The step response is the time and the coordinate dependence of the voltage along the line when an unit-amplitude voltage step is applied to the beginning of the line. A well-known method of obtaining the step response relies on solving the Telegrapher's equations in Laplace domain, applying the boundary conditions and performing the inverse Laplace transform [7]. This method serves well for the lossless distributed RC line [8]-[10] and results in a solution for voltage in the form of infinite series. However, when attempting to solve in a similar way the step-response problem for the RCG line or the RCGW line, one finds that the inverse Laplace transform does not exist in analytical form. Nevertheless, in [11] the transients on RGC lines are analyzed using Laguerre Z-transformation and the solution in the time domain is given in terms of a Laguerre series.

The objective of the current paper is finding the analytical solution for the voltage-step response of the RCGW line. First, we establish the PDE describing the voltage on the RCGW line. Next, we obtain its general solution using the method of separating the variables. Then we solve this PDE for open-ended line of a finite length using three different sets of initial/boundary conditions. Finally we deduce the step response of the RCG line and discuss some related issues of practical interest.

II. Derivation of the PDE

In this section we derive the PDE describing the voltage $u(x,t)$ behavior for the RCGW distributed line depicted in Fig. 1-C. As shown in Fig. 2 the parameters R and C are the resistance of the conductive layer and the capacitance of the dielectric per unit length of the line, respectively. The loss parameters W and G are the transversal conductivities per unit length along the line. We assume that all these parameters are uniform and time-invariant. The voltages and the currents are assumed to be functions of the coordinate along the line and the time, i.e. $i \equiv i(x,t)$, $i_C \equiv i_C(x,t)$, $i_G \equiv i_G(x,t)$, $u \equiv u(x,t)$, and $u_C \equiv u_C(x,t)$.

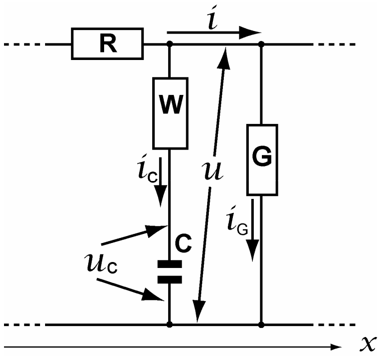

Fig. 2. The meaning of the line parameters, voltages and currents for the distributed RCGW line. For visual clarity the parameters R, C, G, and W, all defined per unit length along the coordinate x, are represented as the discrete elements of a single cell of the line. The voltage checkpoints and the selected positive directions of the currents are indicated by the arrows.

The variation of current along the coordinate x is equal to the sum current through the chains "$C–W$" and "G":

$$\frac{\partial i}{\partial x} = -(i_C + i_G), \tag{1}$$

where $i_G = Gu$.

The variation of voltage along the coordinate x is equal to the voltage drop on resistance R:

$$\frac{\partial u}{\partial x} = -Ri \ . \tag{2}$$

The current i_C charging the capacitance C is given by

$$i_C = C\frac{\partial u_C}{\partial t} \ . \tag{3}$$

Voltages in the chain $C–W$ add up to the line voltage:

$$u = u_C + \frac{i_C}{W} \ . \tag{4}$$

Now we have 4 equations with 4 unknown variables: i, i_C, u_C and u. In order to get the PDE for voltage u we first differentiate (2) by x:

$$\frac{\partial^2 u}{\partial x^2} = -R\frac{\partial i}{\partial x} \ . \tag{5}$$

The substitution of $\frac{\partial i}{\partial x}$ from (5) into (1) gives

$$-\frac{1}{R}\frac{\partial^2 u}{\partial x^2} = -i_C - Gu \ . \tag{6}$$

Differentiation of (4) by t yields

$$\frac{\partial u}{\partial t} = \frac{\partial u_C}{\partial t} + \frac{1}{W}\frac{\partial i_C}{\partial t} \ . \tag{7}$$

The substitution of $\frac{\partial u_C}{\partial t}$ from (7) into (3) gives

$$i_C = C\frac{\partial u}{\partial t} - \frac{C}{W}\frac{\partial i_C}{\partial t} . \tag{8}$$

By substituting the expression of i_C from (6) into (8) we get after some math the final PDE for u in the form:

$$\frac{\partial^3 u}{\partial x^2 \partial t} + \frac{W}{C}\frac{\partial^2 u}{\partial x^2} - R(W+G)\frac{\partial u}{\partial t} - \frac{RGW}{C}u = 0 \ . \tag{9}$$

III. The General Solution for Voltage

Equation (9) can be easily solved by a method called the product method, the separation of variables, or the Fourier's method [12]. We assume that $u \equiv u(x,t)$ has the form of a product

$$u = XT, \tag{10}$$

where $X \equiv X(x)$ and $T \equiv T(t)$. By substituting (10) into (9), dividing the result by XT, and separating the x-dependent and the t-dependent parts we find that

$$\frac{1}{X}\frac{d^2 X}{dx^2} = \frac{(\frac{G}{W}+1)\frac{1}{T}\frac{dT}{dt} - \frac{G}{C}}{\frac{1}{RW}\frac{1}{T}\frac{dT}{dt} + \frac{1}{RC}} \ . \tag{11}$$

Since both sides of this expression are equal, their common value must be a constant. Since only a negative value of this constant, called the separation constant, results in a non-trivial solution we take it to be $-\omega^2$. The separated equations are then

$$\frac{\partial^2}{\partial x^2}X = -\omega^2 X \tag{12}$$

and

$$\frac{dT}{dt} = -\frac{\omega^2 W + RGW}{\omega^2 C + RCW + RCG}T \ . \tag{13}$$

The solutions of the equations (12) and (13) are

$$X = A\sin(\omega x) + B\cos(\omega x) \tag{14}$$

and

$$T = A_1 e^{-\frac{W(\omega^2 + RG)}{C(\omega^2 + RG + RW)}t} \tag{15}$$

respectively, where A, B and A_1 are arbitrary constants.

From (10) we see that the general solution of the PDE (9) for u is

$$u = (A\sin(\omega x) + B\cos(\omega x))e^{-\frac{W(\omega^2 + RG)}{C(\omega^2 + RG + RW)}t} , \tag{16}$$

where the constants A and B already include the constant A_1.

IV. Steady solution for voltage

It is obvious that after a long time under constant input voltage the transients fade out and a steady, time-independent voltage distribution, denoted by $u_{ST}(x)$, forms on the line. For lossless RC line the capacitance becomes eventually uniformly charged up to the source voltage so that $u_{ST}(x) = 1$.

However, for lossy RCGW and RCG lines, a steady current through the distributed resistive network, formed by R and G, still remains. This results in a diminishing voltage along the line. As it can be inferred form Fig. 2, in the steady limit case the current through the capacitance C is ceased, $i_C(x)=0$, and the voltages u and u_C are equalized, $u(x)=u_C(x)=u_{ST}(x)$.

Equation (6) under the assumption $i_C=0$ leads to an ordinary differential equation for steady voltage distribution:

$$\frac{\partial^2}{\partial x^2}u_{ST}(x)=RGu_{ST}(x)\,. \tag{17}$$

Its general solution is

$$u_{ST}(x) = Ae^{\sqrt{RG}x} + Be^{-\sqrt{RG}x}, \tag{18}$$

where A and B are constants, whose values are determined from the boundary conditions. By applying the unit voltage condition at $x=0$, $u_{ST}(0)=1$, and the open end condition, $i_{ST}(L)=0$ or $\frac{\partial}{\partial x}u_{ST}(L)=0$, we find that

$$u_{ST}(x) = \frac{\cosh(\sqrt{RG}(x-L))}{\cosh(\sqrt{RG}L)}\,. \tag{19}$$

V. Applying the boundary conditions

In this section we derive the voltage response for an open-ended RCGW line of a finite length L in the following three cases:

A. shorted input to the line, initially with uniform voltage distribution;
B. unit-voltage-step input to initially discharged line;
C. shorted input to the line, initially with steady voltage distribution.

A. Shorted input to open-ended line with uniform voltage distribution

This case will be considered first, as is seems to be the most intelligible one. We start with establishing the initial condition and the boundary conditions. Initially the voltage is 1V over the entire line:

$$u(x,0)=1\,. \qquad 0<x<L \tag{20}$$

The beginning of the line is short-circuited at $t=0$, i.e. the voltage at $x=0$ is set to zero:

$$u(0,t)=0\,; \qquad t>0\,. \tag{21}$$

The end of the line is open, i.e. $i(L,t)=0$ for $t>0$. The boundary condition for voltage follows from (2):

$$\frac{\partial}{\partial x}u(L,t)=0\,; \qquad t>0\,. \tag{22}$$

As $u(x,t)=X(x)T(t)$ and $T(t)\neq 0$, the boundary conditions similar to (21) and (22) apply also for $X(x)$:

$$X(0) = 0\,; \tag{23}$$

and

$$\frac{\partial}{\partial x}X(L)=0\,. \tag{24}$$

The boundary condition (23) applied to (14) requires that $B = 0$, leaving

$$X(x) = \sin(\omega x)\,. \tag{25}$$

where the arbitrary constant multiplier is omitted for clarity. The boundary condition at $x=L$ now takes the form

$$\frac{\partial}{\partial x}X(L) = \omega\cos(\omega L) = 0\,. \tag{26}$$

A nontrivial solution of this equation exists only if ωL is odd multiple of $\frac{\pi}{2}$ resulting in a series of distinct eigenvalues given by:

$$\omega_n = \frac{(2n-1)\pi}{2L} \qquad n=1,2,... \tag{27}$$

With the eigenvalues in hand we can now specify according to (25) for every n-th eigenvalue the corresponding x-dependent function (the eigenfunction)

$$X_n(x) = \sin(\omega_n x)\,, \tag{28}$$

and according to equation (15) the corresponding t-dependent function

$$T_n(t) = e^{-k_n t}\,, \tag{29}$$

where

$$k_n=\frac{W(\omega_n^2+RG)}{C(\omega_n^2+RG+RW)}\,. \tag{30}$$

The solution of the original problem can be then assembled as a linear combination of the products $X_n(x)T_n(t)$ with (initially arbitrary) coefficients b_n:

$$u(x,t) = \sum_{n=1}^{\infty} b_n X_n(x)T_n(t) = \sum_{n=1}^{\infty} b_n \sin(\omega_n x)e^{-k_n t}\,, \tag{31}$$

where ω_n and k_n are given by (27) and (30), respectively.

The coefficients b_n must be chosen to match the initial condition (20) so that

$$u(x,0)=\sum_{n=1}^{\infty} b_n \sin(\frac{(2n-1)\pi x}{2L})=1\,. \tag{32}$$

Recognizing this equation as the Fourier sine series expansion of $u(x,0)$ gives a recipe for calculation the coefficients b_n. Though the functions $\sin(\frac{(2n-1)\pi}{2L}x)$ differ from those of regular Fourier sine series, $\sin(\frac{n\pi}{L}x)$, they still satisfy the orthonormality condition [12], [15]:

$$\int_0^L \sin(\frac{(2m-1)\pi}{2L}x)\sin(\frac{(2n-1)\pi}{2L}x)dx=\frac{L}{2}\delta_{mn},$$

where

$$\delta_{mn}=\begin{cases}1 & \text{if } m=n\\ 0 & \text{if } m\neq n\end{cases}.$$

Therefore, the Fourier coefficients in (32), b_n matching the initial condition for $u(x,0)$, can be extracted by

$$b_n = \frac{2}{L}\int_0^L \sin(\frac{(2n-1)\pi x}{2L})u(x,0)dx \tag{33}$$

[12]-[15].

Integration according to (33) with $u(x,0)=1$ gives

$$b_n = \frac{4}{(2n-1)\pi}. \tag{34}$$

By substituting the expressions of ω_n and b_n into (31) we can write the final solution for the voltage response satisfying the initial/boundary conditions (20) - (22) in the form:

$$u(x,t) = \sum_{n=1}^{\infty} \frac{4}{\pi(2n-1)}\sin(\frac{(2n-1)\pi}{2L}x)e^{-k_n t}, \tag{35}$$

where the coefficients k_n taken from (30) are given by

$$k_n = \frac{W(4RGL^2+\pi^2(2n-1)^2)}{C(\pi^2(2n-1)^2+4RL^2(G+W))}. \tag{36}$$

For illustration, in Fig. 3 are shown calculated according to (35) instantaneous voltage distributions at different delays after shorting the input of the line. *Ad hoc* values of the line parameters are used.

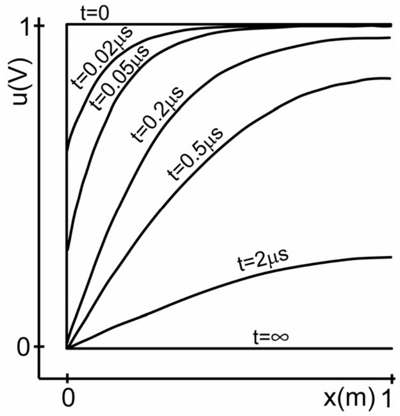

Fig. 3. Response of uniformly charged RCGW line shorted at zero time. The line parameters are $R=2\frac{\Omega}{m}$, $C=2\frac{\mu F}{m}$, $G=0.3\frac{1}{\Omega\cdot m}$, $W=50\frac{1}{\Omega\cdot m}$, $L=1m$.

B. Unit-voltage-step input to discharged open-ended line

The initial and the boundary conditions when a voltage step with the amplitude of 1V is applied at $t=0$ to the input of the initially discharged line are:

$$u(x,0)=0; \qquad 0<x<L \tag{37}$$

$$u(0,t)=1; \qquad t>0 \tag{38}$$

$$\frac{\partial}{\partial x}u(L,t)=0; \qquad t>0. \tag{39}$$

Here the boundary condition (38) is non-homogeneous. Fortunately, using the steady particular solution $u_{ST}(x)$ taken from (19) the original boundary value problem can be transformed to easily solvable homogeneous boundary value problem for another function $w(x,t)$ [12], which is defined as

$$w(x,t) = u(x,t) - u_{ST}(x) \tag{40}$$

and which satisfies PDE (9) as well. Obviously, with $t\to\infty$ $u(x,t)$ approaches to $u_{ST}(x)$ and $w(x,t)$ fades out.

From (37) - (40) we get the initial/boundary conditions for the new function $w(x,t)$:

$$w(x,0)=u(x,0)-u_{ST}(x)=-u_{ST}(x); \qquad 0<x<L \tag{41}$$

$$w(0,t) = u(0,t) - u_{ST}(0) = 0; \qquad t>0 \tag{42}$$

$$\frac{\partial}{\partial x}w(L,t)=\frac{\partial}{\partial x}u(L,t)-\frac{\partial}{\partial x}u_{ST}(L,t)=0. \qquad t>0 \tag{43}$$

The boundary conditions (42) and (43) for $w(x,t)$ appear to be identical to the boundary conditions (21) and (22) for $u(x,t)$ in the case A, respectively. Therefore, the eigenvalues ω_n the eigenfunctions $X_n(x)$ and the time-dependent functions $T_n(t)$ for $w(x,t)$ are the same as for $u(x,t)$ in the case A. As a consequence, the solution for $w(x,t)$ is identical to the solution for $u(x,t)$ given by (31).

In order to find the Fourier coefficients we apply the initial condition (41):

$$w(x,0)=\sum_{n=1}^{\infty} b_n\sin(\frac{(2n-1)\pi x}{2L})=-u_{ST}=-\frac{\cosh(\sqrt{RG}(x-L))}{\cosh(\sqrt{RG}L)}. \tag{44}$$

The coefficients b_n come from (33) where $u(x,0)$ is replaced with $w(x,0)$:

$$b_n=\frac{2}{L}\int_0^L(\sin(\frac{(2n-1)\pi}{2L}x)(-\frac{\cosh(\sqrt{RG}(x-L))}{\cosh(\sqrt{RG}L)}))dx. \tag{45}$$

Performing the integration of (45) we find that

$$b_n = -\frac{4\pi(2n-1)}{\pi^2(2n-1)^2+4RGL^2}. \tag{46}$$

Now we can write the solution for $w(x,t)$, satisfying (41) - (43) in the form

$$w(x,t) = \sum_{n=1}^{\infty} -\frac{4\pi(2n-1)}{\pi^2(2n-1)^2+4RGL^2}\sin(\frac{(2n-1)\pi}{2L}x)e^{-k_n t}. \tag{47}$$

where k_n are given by (36).

The final solution for $u(x,t)$ satisfying the conditions (37) - (39) follows from (40) and is

$$u(x,t)=\sum_{n=1}^{\infty}(-\frac{4\pi(2n-1)}{\pi^2(2n-1)^2+4RGL^2}\sin(\frac{(2n-1)\pi}{2L}x)e^{-k_n t})+\frac{\cosh(\sqrt{RG}(x-L))}{\cosh(\sqrt{RG}L)}. \tag{48}$$

In Fig. 4 are shown calculated according to (48) instantaneous voltage distributions at different delays after connecting the voltage source to the input of the line. The same line parameters as in Fig. 3 are used.

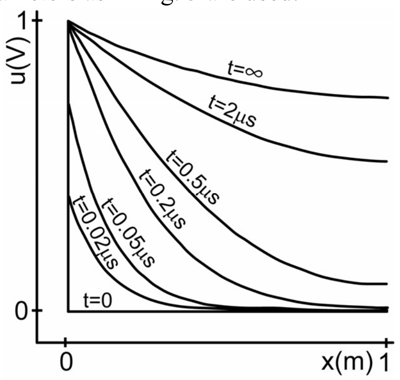

Fig. 4. Response of discharged RCGW line, connected to unit-voltage source at zero time. The line parameters are $R = 2\frac{\Omega}{m}$, $C = 2\frac{\mu F}{m}$, $G = 0.3\frac{1}{\Omega \cdot m}$, $W = 50\frac{1}{\Omega \cdot m}$, $L = 1m$ ·

C. Shorted input to open-ended line with steady voltage distribution

Now we consider an interesting case, where the line has been under unit voltage input until the steady distribution is formed and at $t=0$ the input of the line is shorted.

The initial and boundary conditions for this case are:

$$u(x,0)=\frac{\cosh(\sqrt{RG}(x-L))}{\cosh(\sqrt{RG}L)}; \qquad 0<x<L \tag{49}$$

$$u(0,t)=0; \qquad t>0 \tag{50}$$

$$\frac{\partial}{\partial x}u(L,t); \qquad t>0\ . \tag{51}$$

Similarly to the case A the whole line eventually discharges, so that the steady voltage $u_{ST}(x) = 0$. Since the boundary conditions (50) and (51) are the same as in case A, we can conclude that the solution for $u(x,t)$ given by (31) is still valid.

Compared with case A, the only different parameters are the Fourier coefficients. Calculated according to

$$b_n = \frac{2}{L}\int_0^L \sin(\frac{(2n-1)\pi}{2L}x)\frac{\cosh(\sqrt{RG}(x-L))}{\cosh(\sqrt{RG}L)}dx\,, \tag{52}$$

they are now

$$b_n = \frac{4\pi(2n-1)}{4RGL^2+\pi^2(2n-1)^2}\ . \tag{53}$$

In Fig. 5 are shown calculated according to (31), but with b_n taken from (53), instantaneous voltage distributions at different delays after shorting the input of the line. Again, the same line parameters as in Fig. 3 are used.

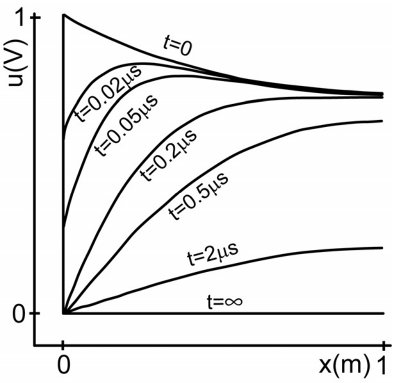

Fig. 5. Response of RCGW line, charged to steady voltage distribution and shorted at zero time. The line parameters are $R = 2\frac{\Omega}{m}$, $C = 2\frac{\mu F}{m}$, $G = 0.3\frac{1}{\Omega \cdot m}$, $W = 50\frac{1}{\Omega \cdot m}$, $L = 1m$ ·

VI. Discussion

In general, the unit-voltage-step response of the RCGW lines can be expressed as

$$u(x,t) = \sum_{n=1}^{\infty}(b_n\varphi(\omega_n x)e^{-k_n t}) + u_{ST}(x)\,, \tag{54}$$

where

$$k_n=\frac{W(\omega_n^2+RG)}{C(\omega_n^2+RG+RW)}\,, \tag{55}$$

and where $\varphi(\omega_n,x)$, ω_n, b_n, and $u_{ST}(x)$ are the eigenfunctions, eigenvalues, Fourier coefficients, and the steady voltage distribution respectively. All these quantities depend on the initial/boundary conditions.

In this paper we presented the solution only for three selected conditions. Naturally, it is possible to define many different initial/boundary conditions that result in an analytical solution. For example, for the short-ended line the eigenfunctions are $\sin(\frac{n\pi}{L}x)$ and the steady voltage distribution is $u_{ST}(x) = -\frac{\sinh(\sqrt{RG}(x-L))}{\sinh(\sqrt{RG}L)}$.

If the conductivity W is shorted, the RCGW line turns into the RCG line. The equation describing the unit-step-voltage response can be deduced from the equation (54) in the limit $W \to \infty$. It turns out that the only term containing W is k_n, that describes the transient part of the voltage. Indeed, the steady voltage distribution $u_{ST}(x)$, the eigenfunctions and the eigenvalues do not depend on W. Moreover, also the Fourier coefficients b_n do not involve W, because they are derived as an integral of the function being a product of an eigenfunction and the initial condition.

When $W \to \infty$, the expression of k_n (55) reduces to

$$k_n = \frac{(\omega_n^2+RG)}{CR}\,, \tag{56}$$

where for our three sets of initial/boundary conditions $\omega_n = \frac{\pi(2n-1)}{2L}$.

In Fig. 6 are shown calculated according to (48), but with k_n taken from (56), instantaneous voltage distributions for a RCG line at different delays after connecting the voltage source to its input. The same line parameters as in Fig. 3, except W, are used.

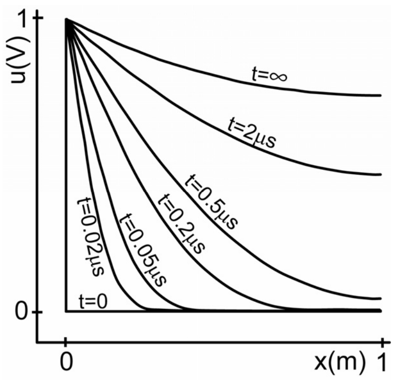

Fig. 6. Response of discharged RCG line connected to unit voltage source at zero time. The line parameters are $R = 2\frac{\Omega}{m}$, $C = 2\frac{\mu F}{m}$, $G = 0.3\frac{1}{\Omega \cdot m}$, $L = 1m$.

By its definition k_n determines the transient behavior of the n-th Fourier component. Thus the parameters k_n can also provide information about the overall charging (or discharging) rate of the line. Since the first eigenvalue $\omega_1 = \frac{\pi}{2L}$ is the smallest one, the corresponding slowest rate k_1 provides an estimate for the lower limit of charging rate of the line with length L.

One can see from (55) that in the case $\omega_n^2 << RG$ $k_n = \frac{WG}{C(G+W)}$, but in the opposite case $\omega_n^2 >> RG$, $k_n = \frac{W}{C}$. Hence for a RCGW line the characteristic changing rate of all transients (including the rate of the lowest transient k_1) is always confined between these limiting values:

$$\frac{WG}{C(G+W)} \le k_n \le \frac{W}{C}.$$

In case of long RCG line, $\omega_1^2 << RG$, $k_1 = \frac{G}{C}$ but in the case of short RCG line $\omega_1^2 >> RG$, $k_1 = \frac{\omega_1^2}{RC} = \frac{\pi^2}{4RCL^2}$.

For a simple RC line the rate of the lowest transient is always given by $k_1 = \frac{\omega_1^2}{RC} = \frac{\pi^2}{4RCL^2} \approx \frac{2.5}{RCL^2}$.

As it can be seen from Figs. 3-5, RCGW line exhibits a stepwise voltage discontinuity at $x=0$ i.e. at the connecting pin of the line with the voltage source. During the first 0.05μs this step decreases from its initial amplitude of 1V down to 0.3V. Indeed, as such a step is formed by high-n eigenfunctions it is expected to decay with a rate given by $\frac{W}{C}$. In the case of $W << G$, the lower limit of k_n, given by $\frac{WG}{C(W+G)}$, approaches to the upper limit of k_n, given by $\frac{W}{C}$. Then the voltage step at $x=0$ endures until the line is charged up to the steady voltage distribution (or discharged down to zero). In the opposite, infinite W, limit i.e. for RCG line the voltage step at $x=0$ disappears just after $t=0$, see Fig. 6. In physical terms for RCG line the capacitance at $x=0$ acquires the source voltage instantaneously, but for RCGW line the conductance W limits the current producing a voltage drop on W and on R at $x=0$ and causing finite charging rate of the capacitance at $x=0$.

In principle, the discontinuity at $x=0$ appears since we neglected the nonzero impedance of the voltage source in our model. This can be taken into account through a modified boundary condition for the beginning of the line. We can write for $x=0$ that

$$u(0,t) = \varepsilon - Si(0,t) = \varepsilon + \frac{S}{R}\frac{\partial}{\partial x}u(0,t), \quad (57)$$

where S is the internal resistance and ε is the electromotive force of the voltage source. The boundary condition at $x=0$ e.g. for the case A takes then the form

$$u(0,t) - \frac{S}{R}\frac{\partial}{\partial x}u(0,t) = 0. \quad (58)$$

The voltage step response can be found as demonstrated in the preceding section. A difficulty with the boundary condition (58) is that each eigenvalue is a solution of a separate transcendental equation and the eigenvalues cannot be longer represented by a common analytical expression.

A variety of mechanical, acoustic, or thermal systems exhibit characteristics very similar to those of electric filters or distributed lines and therefore can be treated by the same mathematical methods [7]. This similarity was put up already by Georg Simon Ohm. He reasoned by establishing an analogy between the Fourier's theory of heat flow and that of electricity. In his scheme the temperature and the voltage correspond as do the heat flow and the electrical current [16]. Following this theory, the voltage along a lossless RC line behaves alike the temperature along a thermally insulated rod. This electro-thermal analogy was utilized by Paschkis and Baker in the middle of the 20-th century. They solved the unsteady unidirectional heat conduction equation with an analog computer composed by a large array of resistors and capacitors [17].

The RCG line is the counterpart of the problem of heat flow along a thermally non-insulated homogeneous rod. Temperature $T(x,t)$ at any point is changing as a result of two phenomena: the diffusion of heat within the rod and the heat flow across the lateral boundaries of the rod. A PDE describing this situation is

$$\frac{\partial T}{\partial t} = \alpha^2 \frac{\partial^2 T}{\partial t^2} - \beta T \quad , \tag{59}$$

where α is the diffusion coefficient and the term βT represents the lateral heat loss [18]. The equivalence between the RCG line and the heat flow is utilized e.g. in [19], where an equation describing the temperature along a heater is obtained in a sophisticated way, using the input and the output impedances of a finite RCG line.

VII. SUMMARY

In this paper we developed one-dimensional model description for distributed lossy RC lines. Two different kind of distributed conductivities were introduced in order to model the loss in the dielectric layer of the line. By constructing and solving a partial differential equation for voltage with properly applied initial/boundary conditions we obtained an analytical expression describing the time-evolution of the voltage distribution on the line after shorting its input (or after applying a voltage step to its input). Finally, we briefly discussed the effect of the conductance W on the voltage response, estimated the overall charging rate of a lossy and a lossless RC line, and pointed out the analogy between a lossy RC line and a linear heat-conducting rod with lateral heat loss.

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