Physics 740: Spring 2007: 03/11/07

P740.11.tex

Waves.

1. In a compressible fluid the *dispersion relation*, (the $\omega - \mathbf{k}$ relation), for wave propagation is

$$\omega^2 = k^2\ c_0^2(\omega) - i\frac{4}{3}k^2\ D_\eta\ \omega - ik^2 c_R^2\ \frac{\frac{k^2 D_\kappa}{\omega}}{1 + \left(\frac{k^2 D_\kappa}{\omega}\right)^2} = k^2\ c_0^2(\omega) - iR_\eta - iR_\kappa, \tag{1}$$

$$c_0^2(\omega) = c_T^2 + c_R^2\ \frac{1}{1 + \left(\frac{k^2 D_\kappa}{\omega}\right)^2}, \tag{2}$$

$$c_R^2 = \frac{\Lambda_\theta \theta_0}{\rho_0 C_V}, \tag{3}$$

where $c_T^2 = (\partial p/\partial \rho)_T$ is the isothermal sound speed and $c_T^2 + c_R^2$ is the adiabatic sound speed, $c_T^2 = (\partial p/\partial \rho)_S$, check this in your Thermo text. See solution to HW3 for further details.

2. A fluid with a free surface, fluid film, an ocean, etc. can generally be regarded as incompressible, $\rho = \rho_0$ and $(\partial p/\partial \rho) = 0$. The forces that drive the fluid are external forces that work on nonequilibrium arrangements of the fluid.

2.a Shallow water. See Fig. 1. The continuity equation is replaced by the rate of change of the height

$$\frac{\partial \delta h}{\partial t} + \nabla \cdot h_0 \mathbf{v} = 0, \tag{4}$$

where usually h_0 can be passed through ∇ if it is independent of x. The equivalent of the Euler equation is an equation for $\mathbf{v}$ (which is independent of z and describes the x, y motion of an entire column of fluid)

$$\frac{\partial \mathbf{v}}{\partial t} = -g\nabla \delta h. \tag{5}$$

These two equations lead to the shallow water wave equation

$$\frac{\partial^2 \delta h}{\partial t^2} = g h_0 \nabla^2 \delta h \tag{6}$$

when h_0 is independent of x. The equilibrium surface of the fluid is a constant height. It is possible that the bottom of the fluid varies, e.g., an ocean. Then h_0 depends on x and we would have

$$\frac{\partial^2 \delta h}{\partial t^2} = g\nabla \cdot (h_0 \nabla \delta h) \tag{7}$$

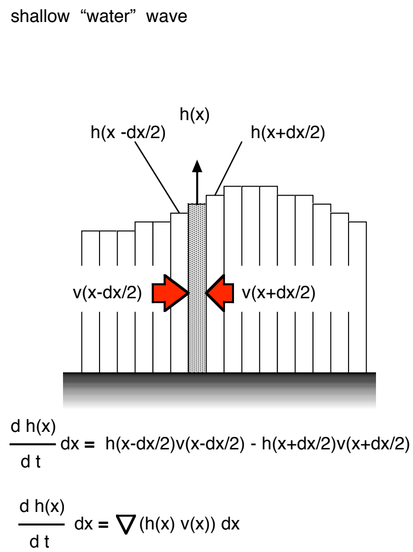

FIG. 1: Conservation of fluid for shallow water.

2.b Deep water. See Fig. 2. In the deep water case we have to retreat to a more formal treatment of the fluid. The variable that describes the displacement of the surface from its undisturbed fluid location is $\zeta(x, y, t)$. At the bottom of the fluid we have the usual boundary condition $\mathbf{v} \cdot \mathbf{n} = 0$. In the interior of the fluid we have

$$\nabla^2 \phi = 0. \tag{8}$$

At the top of the fluid we have to relate ζ to ϕ, the pressure, the gravitational force, etc. To do this we write out the Bernoulli equation is general form. From the manipulations in **P740.9.tex** we had the Euler equation in the form

$$\frac{\partial \mathbf{v}}{\partial t} - \mathbf{v} \times (\nabla \times \mathbf{v}) = -\nabla \left(\frac{P}{\rho_0} + \frac{1}{2} v^2 + U \right), \tag{9}$$

where ρ_0 a constant, $\nabla \times \mathbf{v} = 0$, $\mathbf{v} = \nabla \phi$ and $-\nabla U$ gives the force per unit mass. Thus

$$\frac{\partial \phi}{\partial t} + \frac{P}{\rho_0} + \frac{1}{2} v^2 + U = C. \tag{10}$$

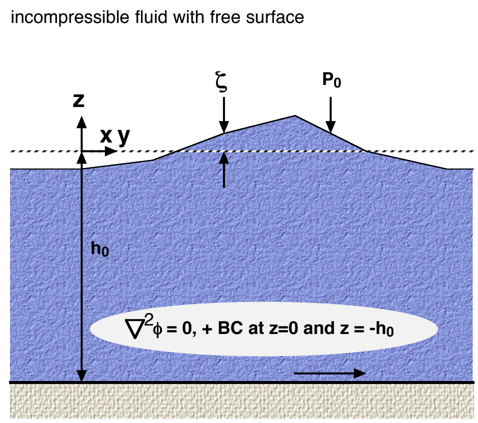

FIG. 2: Deep water.

On the free surface of the fluid

1. the term in v^2 is second order and can be dropped,
2. $P = P_0$,
3. $U = g\zeta$,
4. choose $\phi = \hat{\phi} - (C + P_0/\rho_0)t$ to get rid of the constant factors.

Then

$$\frac{\partial \hat{\phi}}{\partial t} + g\zeta = 0 \tag{11}$$

We also have

$$v_z = \frac{\partial \hat{\phi}}{\partial z} = \frac{\partial \zeta}{\partial t}. \tag{12}$$

Thus on the fluid surface we have

$$\frac{\partial \hat{\phi}}{\partial z} = -\frac{1}{g}\frac{\partial^2 \hat{\phi}}{\partial t^2}. \tag{13}$$

Summary. For an incompressible fluid with a free surface

1. $\nabla^2\hat{\phi} = 0$ in the interior of the fluid,

2. $\mathbf{n} \cdot \mathbf{v} = 0$ on the bottom of fluid,

3. $$\frac{\partial \hat{\phi}}{\partial z} = -\frac{1}{g}\frac{\partial^2 \hat{\phi}}{\partial t^2}, \tag{14}$$

   at the top of the fluid.

Example. A simple fluid of uniform depth h_0. Choose

$$\hat{\phi} \equiv \phi = Z(z)X(x)e_{-i\omega t}. \tag{15}$$

Then

$$\frac{1}{Z}\frac{d^2 Z}{dz^2} + \frac{1}{X}\frac{d^2 X}{dx^2} = \kappa^2 - \kappa^2 = 0, \tag{16}$$

$$X \propto \sin\kappa x, \tag{17}$$

$$Z \propto \cosh\kappa(h_0 + z), \tag{18}$$

where this choice of Z is such that $dZ/dz \propto \sinh\kappa(h_0 + z) = 0$ at $z = -h_0$. From Eq. (14) find

$$\omega^2 = g\kappa \tanh\kappa h_0, \tag{19}$$

the dispersion relation for water waves.

1. shallow water, $\kappa h_0 \ll 1$,

   $$\omega^2 = g h_0 \kappa^2. \tag{20}$$

2. deep water, $\kappa h_0 \gg 1$,

   $$\omega^2 = g\kappa. \tag{21}$$